TRANSPOSITIONS

LOG. (*coming across Math in the passage of the L. E. J. Brouwer Institute*) Hallo Math! Still pondering about school geometry?

MATH. What is wrong with that? Years ago I had to solve many problems in Euclidean geometry, and some of them were quite tough. Do you know Morley's triangle?

LOG. I do, and I agree that the proofs of this theorem are all pretty complicated. Maybe they can be made more perspicuous by discerning the applications of redescriptions and substitutions?

MATH. I will later try to do that, But on the moment I am also doing geometry, although a different branch of it. I am now working on finite geometries.

LOG. Is there still something to discover?

MATH. Certainly, but it is also a field in which results can be booked by applications of another procedure than redescription and substitution, namely transposition. Shall I explain it to you?

LOG. With pleasure!

MATH. Then I will ask Comp to join us, because he helped me in finding some conspicuous results. (*He uses his mobile telephone and after a while Comp appears. Together they go into Math's room.*)

COMP. Shall I give an introduction into the subject of transpositions? After all, Math's work started with a typical computational problem, namely getting a competition scheme for a chess tournament.

MATH. I am glad that you will start! Without your help I would not have come so far.

COMP. Suppose we want to determine a competition scheme for an even number of players in such a way that every two players encounter each other in exactly one round, in which all players participate. This problem can easily be solved by a computer for 6 players, indicated with the numbers 1, 2, 3, 4, 5, 6. We can simply use a standard search procedure (with lexicographical ordering and depth first search). The solution below shows the notation, and if necessary also the structure, of a ‘tournament’ consisting of ‘rounds’, and ‘rounds’ consisting of ‘games’. Look (Figure 1):

12 34 56
13 25 46
14 26 35
15 24 36
16 23 45

Figure 1

The solution could also be found without a computer. However, human beings are not proficient in this procedure for larger numbers, and even computer programs may fail for more than 200 players, as I saw myself. But this is not as terrible as it may look, for there is also a simple solution, accomplished by a what Math calls a transposition. It was given by Kraitchik and Math and I call it ‘Kraitchik’s solution’. The idea can be demonstrated for 6 players as follows. Replace or represent each number by a point in such a way that 1 is represented by the centre of a circle, and the other numbers by points, regularly distributed on the circle. Then the first round is read off from the figure by taking the line 12 and its perpendiculars 36 and 45, the second round by 13 and its perpendiculars 42 and 56, and so on (Figure 2).

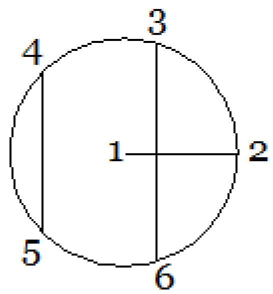

Figure 2

It is immediately clear that this method can be applied to every even number of players.

LOG. That is indeed amazing! But can you tell me which conditions make this transposition possible?

COMP. I think that this can be better explained by Math.

MATH. Is it easy to see that two 'simple' elements – in this case the players, or the numbers – determine exactly one 'complex' element – in this case the games, or the pairs of numbers.

LOG. I see, and this suggests that a similar procedure will also be successful in the task of finding arithmetical models of elementary geometrical axiomatic systems about points and lines, on the condition that two distinct points determine a straight line, or, in other words, for any two points there is exactly one (straight) line containing them both. Let me explain.

MATH. This was also my idea, and therefore I started with simple geometrical theories A(n) in which the following axioms hold:

1. For every two distinct points, there is exactly one line containing them both.
2. Through a point not on a given line there is exactly one line that does not meet the given line.
3. Not all points are on the same line.
4. There exists at least one line.
5. Every line contains exactly n points.

LOG. I know a model of this system when every line contains exactly two points (Figure 3):

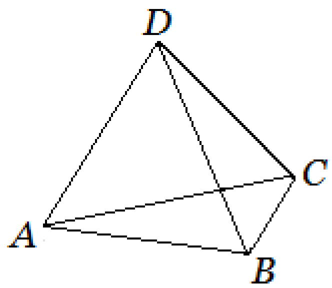

Figure 3

But I have never seen a model in which every line contains exactly three points. I understand that your procedure will produce such a model?

MATH. It will, but let us first make a transposition in the sense that points are replaces by numbers and lines by arrays of numbers. In your example this would give something such as

12 23 34
13 24
14

Figure 4

COMP. This suggests that it is again a combinatorial problem.

MATH. This was indeed the way in which, a long time ago, John Wesley Young solved the problem for $n = 3$. He did it by a certain reasoning process, with the following result (Figure 5):

123 246 349 478 569
145 258 357
167 279 368
189

Figure 5

COMP. This solution can also be achieved by a brute force method, using the lexicographical ordering and depth first search.

MATH. That is difficult for human beings, as soon as larger values for n are taken. But we take advantage of the presence of the above-mentioned condition that three numbers determine exactly one array. Therefore we try the hypothesis that solutions can also, and more easily, be found after a transposition into a domain consisting of a circle with center 1 and containing the other eight points at equal distances from each other. That the answer is positive for $n = 3$ can be seen from the following perspicuous representation, in which *three* ‘lines’ are pictured, whereas the other nine ‘lines’ are found by rotations (Figure 6):

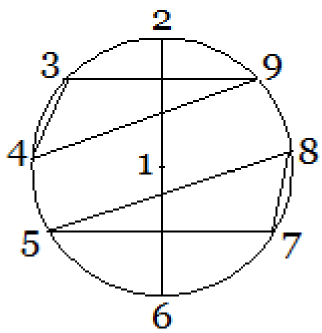

Figure 6

Apparently the resulting ‘arithmetical’ solution is as follows (Figure 7):

126	349	578
137	452	689
148	563	792
159	674	823

Figure 7

LOG. That is interesting. As soon as the 'geometrical' solution has been found, it is not necessary any more to carry out the rotations. The successive lines are determined by cyclical permutations.

MATH. In any case, we have found a perspicuous model of the axiom system A(3), and it is clear that we want to find a similar picture as Figure 6 for axiom system A(4), in which axiom 5 says that every line contains exactly 4 points.

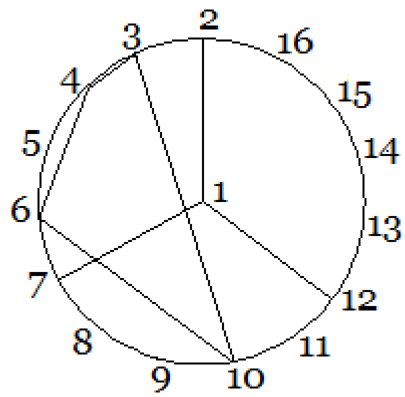

Figure 8

LOG. Do I understand you well, as I conclude that this figure shows the two 'lines', 1 2 7 12, and 3 4 56 10 and that the third and fourth 'lines' are found by rotations of the second 'line' over 120 degrees?

MATH. That is right. The usual cyclical permutations finish the job. Their result is the following 'arithmetical' model of the axiom system A(4) (Figure 9):

1 2 7 12	3 4 6 10	8 9 11 15	13 14 16 5
1 3 8 13	4 5 7 11	9 10 12 16	14 15 2 6
1 4 9 14	5 6 8 12	10 11 13 2	15 16 3 7
1 5 10 15	6 7 9 13	11 12 14 3	16 2 4 8
1 6 11 16	7 8 10 14	12 13 15 4	2 3 5 9

Figure 9

Of course, this is not the end of the story. One would also like to find a model for the axiom system A(5), but the above solution does not give sufficient indications about how to find a solution for this problem of A(5) in a systematic way. Therefore I switch over to the even simpler geometrical theories P(*n*), in which the following axioms hold:

1. For every two distinct points, there is exactly one line containing them both.
2. For every two distinct lines, there is exactly one point contained by both.
3. Not all points are on the same line.
4. There exists at least one line.
5. Every line contains exactly *n* points.

LOG. I know also a model of this system when every line contains exactly two points. It is the Fano's famous projective (Figure 10):

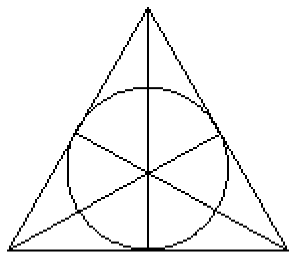

Figure 10

COMP. That is nice, and it is easy to derive an arithmetical model from this picture. However, we started with an arithmetical model for P(3) with a standard brute force procedure (Figure 11):

123	246	347
145	257	356
167		

Figure 11

The similarity with the foregoing cases suggests that here also a transposition is possible, or, more precisely, that a perspicuous representation can be found in the form of a circle, so that rotations can do the work. It appears that we succeed as soon as we place all points on a circle, and choose a triangle in such a way that it has exactly one point in common with each of its rotations around the center of the circle (Figure 12):

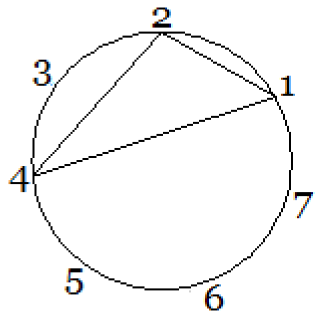

Figure 12

MATH. Similarly, a model for P(4) was found with a suitable quadrangle (Figure 13):

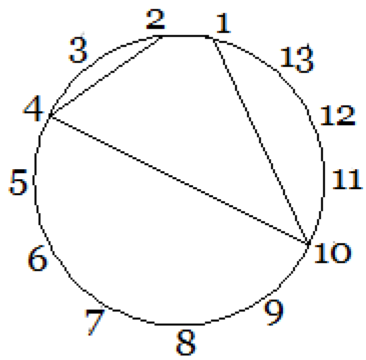

Figure 13

Obviously, the corresponding numerical model begins with 1 2 4 10, followed by 2 3 5 11, and so on, until 13 1 3 9 has been reached. But now the question can be asked what is so special about the division of the points on the circle, that every two of these quadrangles have exactly one point in common. The answer is that the successive distances of the vertices of the quadrangle are, respectively, 1, 2, 6 and 4, and this implies that the sums of adjacent distances on the circle are all different. In arithmetical terms: 13 = 1 + 2 + 6 + 4 is such that each number under 13 appears just once as a partial sum in this equation, in the sense that 1 = 1, 2 = 2, 3 = 1 + 2, 4 = 4, 5 = 4 + 1, 6 = 6, 7 = 4 + 1 + 2, 8 = 2 + 6, 9 = 1 + 2 + 6, 10 = 6 + 4, 11 = 6 + 4 + 1 and 12 = 2 + 6 + 4. As soon as this is seen, the construction of models for P(5) and P(6) is easy. Without much effort, the partitions 21 = 1 + 3 + 10 + 2 + 5 and 31 = 1 + 14 + 4 + 2 + 3 + 7 can be found.

LOG. This is a remarkable result! Apparently the hypothesis can be formulated that the problem of finding models for the axiom systems A(n) can also be solved by a transposition to the arithmetical domain in which suitable partitions are found.

MATH. Inspection of the geometrical model for A(3) – Figure 6 – yields the equation 8 = 1 + 5 + 2, in which all numbers under 8 except 4 occur as a partial sum. Similarly, the picture for A(4) – Figure 8 – gives 15 = 1 + 2 + 4 + 8. But now we know how to proceed in the as yet unsolved case of A(5), if the search for a suitable pentagon would get stuck: by a transposition! Although I still found a solution by means of the geometrical representation, resulting in thre fpllowing picture, the 'modern' way of finding a model-forming partition is leaving the tedious work to the computer.

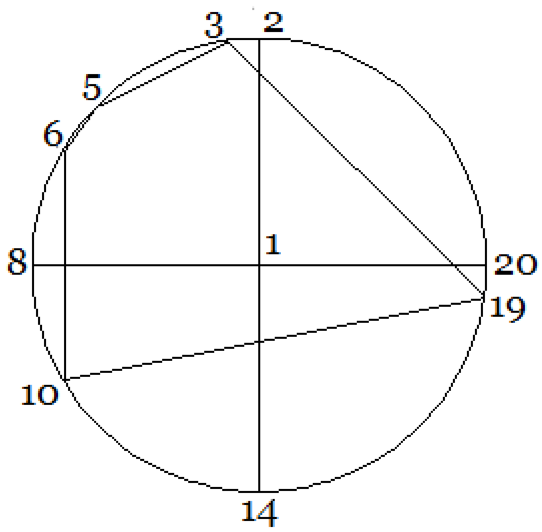

Figure 14

So after I had found the above solutions, I asked Comp for help.

COMP. I wrote two computer programs, one for the partition problem that resulted after the three (!) transpositions of the problem of finding a model for axiom systems A(n), and another for the same problem for axiom systems P(n). It appeared that the partition problem for A(5) has two different solutions. The first is given by the equation 24 = 1 + 2 + 8 + 9 + 4, which can already be read from Figure 10, and the second by the equation 24 = 1 + 3 + 5 + 2 + 13. However, the partition problem for A(6) appeared to have no solution at all, and this means that this axiom system has no models. On the other hand, the partition problem for A(7) has four different solutions, and the one for A(8) has three.

Similar results were found for axiom systems P(n). My program showed that the partition problem has two different solutions for P(4), for P(5) only one, for P(6) five, but for P(7) none. It follows that the axiom system for P(7) has no models.

MATH. There is no need to pursue this subject further, since it may be assumed that such results are corollaries of theorems in affine or projective geometry. My purpose was to show how productive problem solving could arise from a series of transpositions, from geometrical representations to arithmetical representations, from these arithmetical representations to other geometrical representations, and from these geometrical representations to other arithmetical representations. Mathematicians and computer scientists can help each other in finding solutions to problems, depending on the types of representation each of them can manage best.

LOG. Your mathematical approach is rather intuitive, whereas I stress the role of logical reasoning in problem solving. What do you think of that?

MATH. It is true that my successes depend to a large extent on the role of my representations. We have seen that the choice of a particular medium for a certain problem has been guided by the success that was achieved with it in the case of a similar problem. This was an example of reasoning by analogy. But the choice of another medium is one thing, and the selection of certain representations in a chosen medium another. In Kraitchik's solution of the competition problem, a circle plus center was selected instead of only a circle. Then a particular representation, consisting of a diameter with perpendiculars, gave the key to the general solution.

It is easily seen that the choice of a circle without center would have yielded simple solutions for the competition problem with 6, 8, and 10 players. One has only to find suitable combinations of sides and diagonals of (regular) polygons, but their different structures already show that this is not the right road to a general solution. So intuitive guesses are not a guarantee for success. This depends on the elaboration of the ideas, and that requires much logical reasoning. Moreover writing computer programs requires it too. Yet it is possible that the possibility of a transposition is, in a sense, derived from solutions found in a certain medium. Let me explain.

Cosinder the numerical solution of the tournamnet problem for six players – Figure 1. We can transpose each of the rounds to the figure of a circle with center 1 and the other numbers regularly divided over the circle, and then conclude that the third round, 14 26 35, leads to a configuration that leaves nothing to be desired (Figure 15):

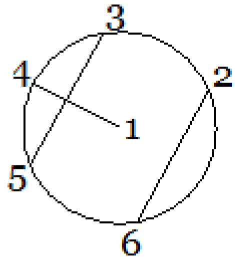

Figure 15

That this picture contains more or less a complete solution of the composition problem for six players, if not for every even number of players, is clear to any well-trained mathematician. Nevertheless it does not require a great effort to find this figure, at least if the above procedure is followed. As soon as the right transpositions are carried out, the solution of the problem is ready to hand. The same holds for the preceding examples. But we have also seen that the choice of promising transpositions was guided by the idea that they were effective under similar conditions. Thus there are two psychologically relevant aspects of the procedure of transposition: first, the idea that another medium might further the solution, second, the insight that the solution is contained in a particular representation. But there is also a philosophical relevant aspect of
transposition.

LOG. What do you mean by that? You know that I consider philosophy a waste of time. It bestows a premium to vagueness and history has shown that it may lead to an anti-scientific attitude.

MATH. That is not what I mean. My point is only that mathematicians have a certain 'freedom' in choosing the medium of their liking. Admittedly the notion of freedom is vague, but philosophy should elucidate it within a definite context, in my case the study of finite geometries.

LOG. OK, go ahead.

MATH. The given geometrical figures show, in a sense, how models of finite geometries can arise, but they are themselves not complete models. It seems that performing the rotations would result in pictures that are not perspicuous anymore, as this is the case for A(2) (Figure 16) and P(2) (Figure 17):

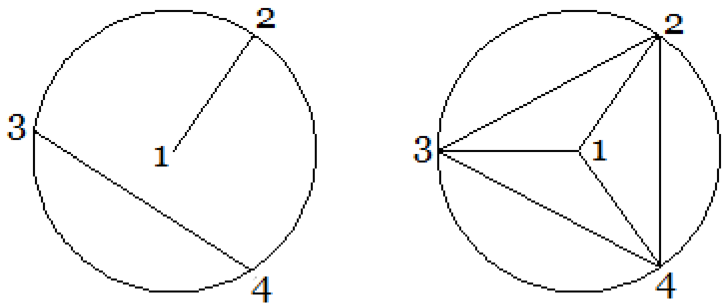

Figure 16

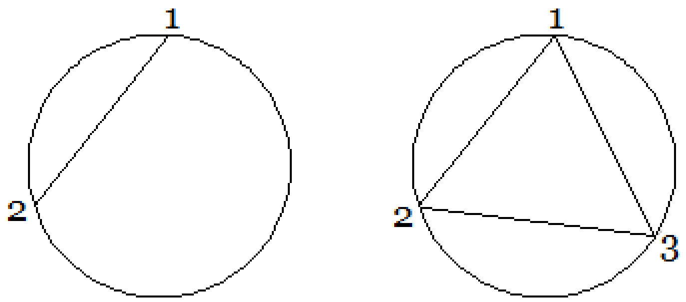

Figure 17

This is, of course, nothing new. At most we may notice that the complete model of A(2), as pictured in Figure 16, can be seen as a three-dimensional figure that is in principle not different from your picture – Figure 3 again:

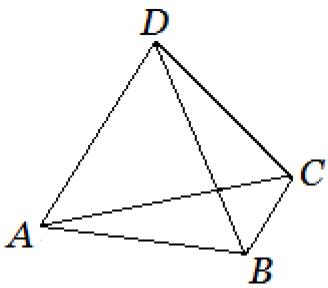

Figure 3

LOG. Fano's projective plane is indeed a perspicuous model of P(3). But it is a pity that we have as yet no perspicuous picture of a model of A(3).

MATH. Yet there is a possibility to depict a complete model of A(3) too, if one allows that every point gets two 'locations'. The same holds for P(3), which thereby gets a new perspicuous representation. The results are sketched in the following pictures (Figure 18 and Figure 19):

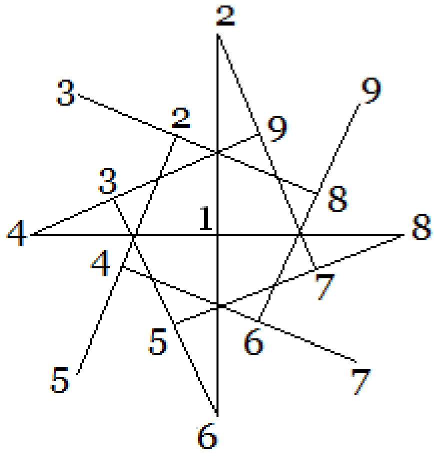

Figure 18

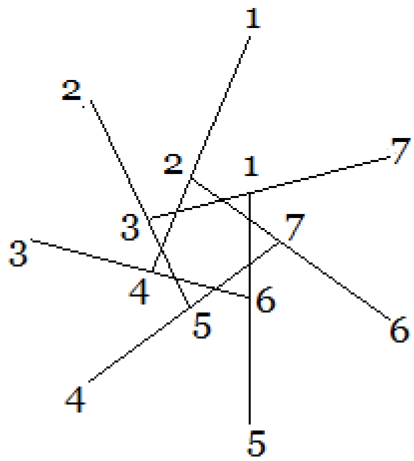

Figure 19

But now there is a remarkable difference between the last picture that shows the model of P(3), and Fano's famous 'projective plane' – Figure 10 again.

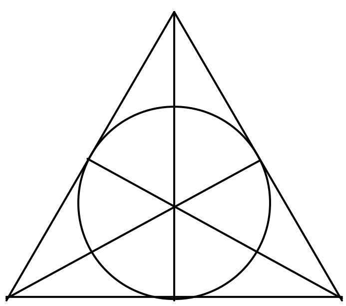

Figure 10

Figure 17 shows the cyclical structure of the model, whereas the form of Figure 8 suggests something quite different.

It is also possible to construct rotation symmetrical representations of models for A(4) and P(4), provided that every point is given three locations. In order to do this in an easy way for P(4), we start with the line containing the points 1, 5, 6 and 8 and continue with the line containing the points 3, 7, 8 and 10 as follows (Figure 20):

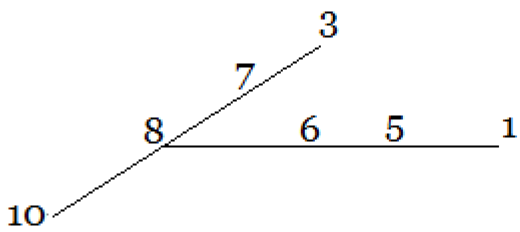

Figure 20

Then we rotate this figure in such a way that the first line 1, 5, 6, 8 coincides with the second line and the second line produces a third line with the points 5, 9, 10 and 12. By repeating this procedure, we successively get the other lines 7, 11, 12, 1; 9, 13, 1, 3; 11, 2, 3, 5; 13, 4, 5, 7; 2, 6, 7, 9; 4, 8, 9, 11; 6, 10, 11, 13; 8, 12, 13, 2; 10, 1, 2, 4 and 12, 3, 4, 6. This results in a figure that is similar to Figure 17 and has also a nice rotational symmetry.

LOG. It is clear. There is no need to pursue the subject further. But the fact that points can get more ‘locations’ is new to me. What is your ‘philosophy’ about it?

MATH. It shows that it makes no sense to speak of ‘points’ as if the ‘nature’ of such mathematical objects is ‘fixed’, let alone ‘predetermined’: it is the creative mathematician who decides how to represent mathematical objects. He is free to depict them in the medium of his liking, and when he wants to represent one and the same point by more than one ‘dots’, he may do this and describe it according to his preference. He is not interested in the nature of mathematical entities, but only in the nature of representations of mathematical entities, as long as they are subservient to his aim, that is, solving mathematical problems and requiring insight into the obtained solutions.

LOG. This reminds me of a remark made by one of my teachers, Heyting, at one of his classes on Intuitonist mathematics. He said that philosophical discussions on ‘the nature of mathematical entities’ are only relevant if the adopted points of view influence the way in which mathematicians are actually reasoning.

MATH. I have heard that too. But what do practicing mathematicians themselves think of the question what mathematics is about? They do not subscribe to the (ironical) view, once formulated by Russell, that 'mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true' Neither do they hold that mathematical entities 'participate' in a 'sphere' called 'logical reality', as E. W. Beth thought. Mathematicians simply do not pursue the question so deeply; their 'objects' are numbers, points, functions, groups, etc. However, there is a kind of relativity involved: a mathematical theory is not *per se* about, say, points or 'geometrical objects', for it may happen that a certain problem, allegedly about such things, can be better solved by imagining that it is about other things such as numbers or 'arithmetical' objects, and conversely.

I conclude that 'transpositions' can have an important heuristic significance. Switching from one 'domain' to another, more perspicuous 'field of activity', may facilitate the mathematical problem-solving process, accordingly as either the mathematician's 'intuitive' skills, or the computer's 'digital' powers can be better exploited in it. This has beer abundantly demonstrated by solutions to the problem of finding models for finite affine and projective geometries.

LOG. Your argument supports the view that the 'nature of the mathematical objects' may indeed be relevant to the way in which mathematicians are actually reasoning, though in a mundane interpretation which is totally different from its philosophical meaning.

COMP. I am glad that you have been brought down to earth. It reminds me that I have still a lot to do.

LOG. I will join you, Comp. (*They greet Math and leave his room.*)