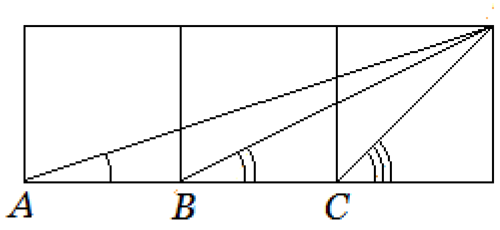
Three squares

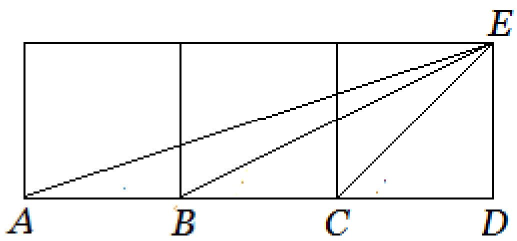
(Lyber Katz)

Original formulation:



Prove that $\angle A + \angle B = \angle C$

Euclidean formulation:



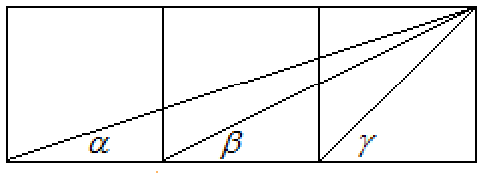
Prove that $\angle EAD + \angle EBD = \angle ECD$

References

Martin Gardner, *Mathematical Circus*. Harmondsworth, Middlesex, England: Penguin Books, 1982 (reprinted), p. 125
http://maa.org/pubs/focus/MH_092010_GardnerEve.pdf
http://archive.org/details/LyberKatz29nov2010YiddishBookCenter

Hints

Consider the following formulation:



Prove that $\alpha + \beta = \gamma$

There are at least two different strategies to start with:

(1)

Regard the angle γ as an exterior angle of a suitable triangle, and draw a conclusion about its as yet unknown interior angle. Find another promising triangle! Think of relevant central theorems!

(2)

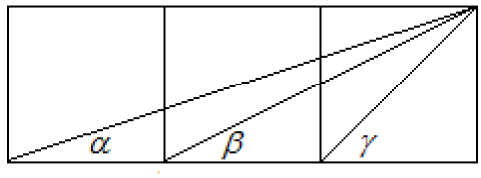
Form an angle $\alpha + \beta$, using one of the given angles. Make it a part of a promising triangle! Think of relevant central theorems!

(3)

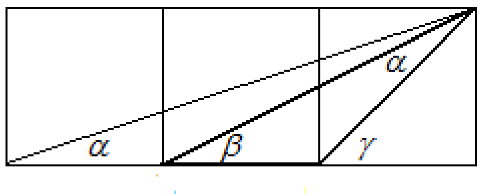
If this is not helpful, then try to give a goniometric proof. Think of the central relevant theorem!

Solutions

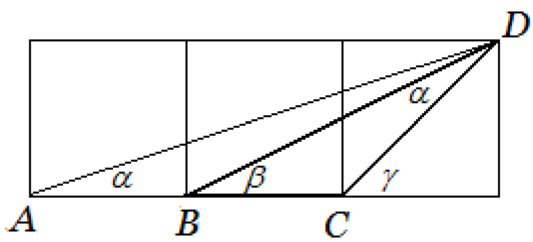
(1)



If $\gamma = \alpha + \beta$, then the following marked triangle must have an angle α, according to the exterior angle theorem:



This means that the triangle *BCD* must be equal to the triangle *DCA*:



They already have one angle in common, so we must prove that

$$BC : CD = DC : CA$$

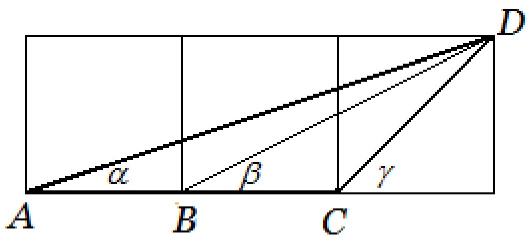
in order to apply the corresponding central theorem of equal triangles.
Let BC be 1, then $CD = \sqrt{2}$ and $CA = 2$, and the desired result follows:

$$1 : \sqrt{2} = \sqrt{2} : 2$$

So the triangle *BCD* is indeed equal to the triangle *DCA*.
Therefore $\angle CDB = \angle CAD = \alpha$, and the exterior angle theorem gives

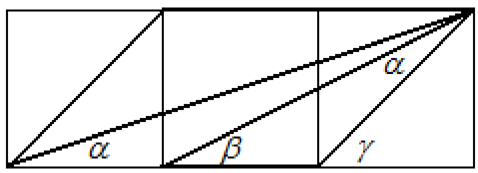
$$\gamma = \alpha + \beta$$

Remark: The angle γ is also an exterior angle of the triangle ACD:



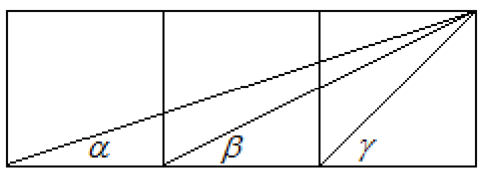
Then it must be proved that $\angle ADC = \angle DBC = \beta$.

A variant of these proofs makes use of another triangle. It requires an auxiliary line segment:

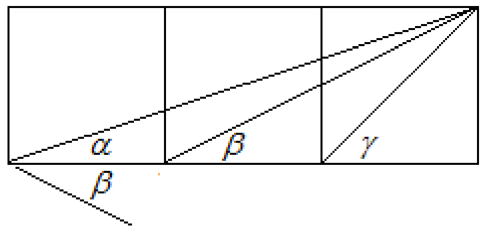


These two similar triangles are more easily distinguishable than the similar triangles in the above proof. However, this way of solving the problem presupposes the knowledge of similarity and the Pythagorean theorem. That this is not at all necessary, is shown by the next solution.

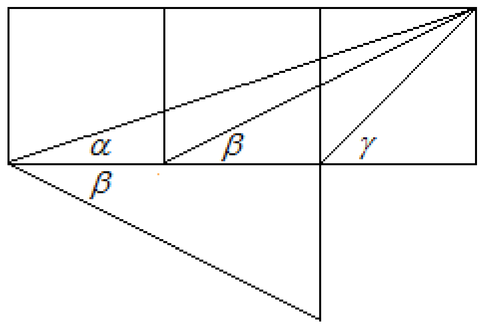
(2)



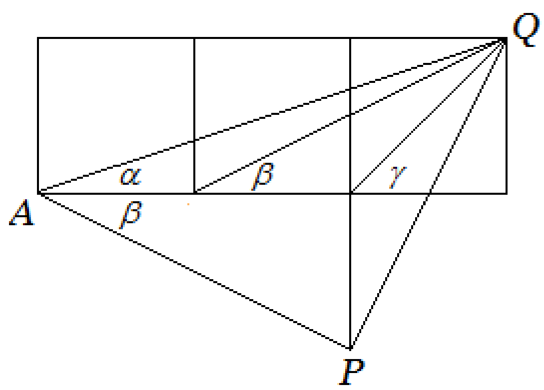
An angle $\alpha + \beta$ can be obtained by putting an angle β next to the original angle α:



Is there a promising triangle which contains this angle β? Of course, there is one:



With only one more auxiliary line segment, we get still another triangle, congruent with this triangle:



The resulting triangle *APQ* is not only an isosceles triangle, but it has a right angle $\angle APQ$ too. It follows that $\alpha + \beta$ is half of a right angle, and therefore

$$\alpha + \beta = \gamma$$

This solution requires very little knowledge!

Instead of putting an angle β 'below' the angle α, an angle α can be put alongside the original angle β. The elaboration is left to the reader.

(3)

There is also a goniometric solution, but it requires that the sum formula for the tangens is known:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Moreover, to me it looks like a trick ... The proof is left to the reader!

Addendum

(4) A proof, given by Martin Gardner, makes use of two similar triangles *BDF* and *AGF*: