THE NINTH POWER OF THIRTEEN

LOG. (*Enters Math's room*) Do you have time for me, Math?

MATH. For you always!

LOG. Because I was thinking about representations as a sum of two squares, I consulted Wolfram Alpha for such representations of powers of 13. Here, I noticed that 13^8 was the highest power for which these representations were given. This worried me, and therefore I ask you for help. Can you find at least one representation of 13^9 as a sum of two squares?

MATH. Of course I can! I will need only two elementary theorems: (*He goes to the blackboard*)

(I) If a natural number n is a sum of two squares, then n^2 is also a sum of two squares

(II) The product of two different sums of two squares can be written in two ways as the sum of two squares

LOG. Your first theorem is almost trivial for everyone who knows the construction of Pythagorean triangles: (*She also goes to the blackboard*)

$$n = p^2 + q^2$$
$$n^2 = 2pq + (p^2 - q^2)^2$$

MATH. The second theorem can also be easily proved. When I was still a high school teacher, I gave it as an exercise to my students.

LOG. Nevertheless it is new to me, but let me see:

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= a^2c^2 + b^2d^2 + 2abcd - 2abcd + a^2d^2 + b^2c^2 \\ &= (ac + bd)^2 + (ad - bc)^2 \\ &= (ac - bd)^2 + (ad + bc)^2\end{aligned}$$

MATH. Quite right! Any idea how to find a representation of 13^9?

LOG. Given that $13 = 3^2 + 2^2$, we can successively apply your theorems to get a representation of 13^2, 13^3, 13^6, and, finally of 13^9.

MATH. I will do it:

$13 = 3^2 + 2^2$

$13^2 = (2\cdot3\cdot2)^2 + (3^2 - 2^2)$

$13^2 = 12^2 + 5^2$

$13^3 = (3\cdot12 + 2\cdot5)^2 + (2\cdot12 - 3\cdot5)^2$

$13^3 = 46^2 + 9^2$

$13^6 = (2\cdot46\cdot9)^2 + (46^2 - 9^2)^2$

$13^6 = 828^2 + 2035^2$

$13^9 = (46\cdot828 + 9\cdot2035)^2 + (46\cdot2035 - 9\cdot828)^2$

(*He takes his pocket calculator*)

$13^9 = 56403^2 + 86158^2$

Here you are!

LOG. Thank you! It's one thing less to worry about! Till tonight! (*She leaves Math's room)*

(*Now Math consults Wolfram Alpha too; see here what he finds*)

$10\,604\,499\,373 = 13^9$ is a perfect 9^{th} power.

10604499373 is the hypotenuse of a primitive Pythagorean triple:

$10\,604\,499\,373^2 = 4\,241\,902\,555^2 + 9\,719\,139\,348^2$

(*He checks this immediately with the help of his first theorem*)

$13^{18} = (2\cdot86158\cdot56403)^2 + (86158^2 - 56403^2)^2$

$13^{18} = 9719139348^2 + 4241902555^2$

(*Then he goes to Log's room*)

MATH. Did you see that Wolfram mentioned that the square of 13^9 is the hypotenuse of a primitive Pythagorean triple?

LOG. Yes, but what does this mean?

MATH. That he knew that 13^9 has the representation we found, because this property of the square of 13^9 follows directly from it!

LOG. So Wolfram knows more than he tells us!

(*They part laughing*)