SQUANGLES

COMP. Good morning Math, what can I do for you?

MATH. Nothing at all, but I want to show you something.

COMP. Go ahead, my computer can wait.

MATH. Do you remember my Transpositional Tricks?

COMP. Of course, I liked your geometrical pictures, do you have more of those nice things?

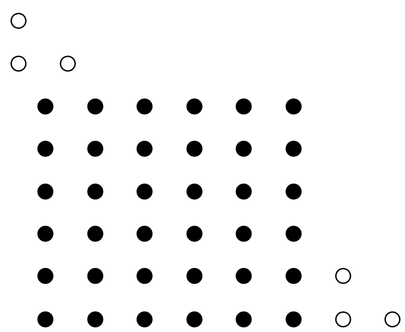
MATH. In a way, yes, but let me first of all state the problem. You know that I found a beautiful representation of the equation 666 = 441 + 225, which I analyzed as an equation between the triangular number of a square – 36 – and the squares of two successive triangular numbers – 15 and 21. Well, when I looked at the tables of the squares and the triangular numbers, I noticed that 36 occurs in both of them and so I wondered whether there are more squares that are also triangles. By the way, I call such numbers squangles.

COMP. Was this your problem? Wait a moment and my computer will give you the answer!

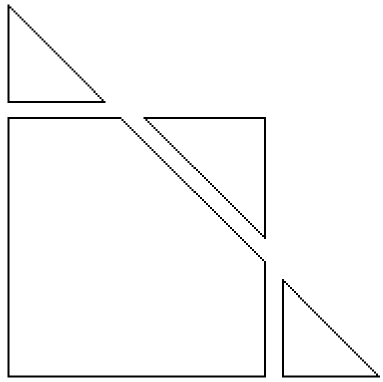
MATH. Stop, this is not a computational problem. I wanted to have insight into the mathematics behind the equation s(m) = t(n), I am not interested in the
solutions as such.

COMP. I see, tell me what you've found.

MATH. (*walks to the blackboard*) First of all I drew the following picture of the equation s(6) = t(8):



I noticed that two triangles of 2 are added to the square of 6 in order to get the triangle of 8 whereas one triangle of 3 is removed. My intuitive inference was that in general each squangle can in this way be transformed, that is, by replacing a triangle number by two congruent triangles. The following figure or scheme pictures this result:



My promising conclusion was that a table with triangular numbers and their doubles would lead me faster to more examples of squangles.

COMP. So you used a computer for the construction of such a table?

MATH. Not at all. I only calculated as much values of t(n) and 2t(n) as was necessary to find the next squangle after 36. Here I will do it again and mark the equal outcomes:

n	t(n)	2t(n)
0	**<u>0</u>**	**<u>0</u>**
1	**1**	**2**
2	**3**	**<u>6</u>**
3	**<u>6</u>**	**12**
4	**10**	**20**
5	**15**	**30**
6	**21**	**42**
7	**28**	**56**
8	**36**	**72**
9	**45**	**90**
10	**55**	**110**
11	**66**	**132**
12	**78**	**156**
13	**91**	**182**
14	**105**	**<u>210</u>**

15	**120**	**240**
16	**136**	**272**
17	**153**	**306**
18	**171**	**342**
19	**190**	**380**
20	**<u>210</u>**	**420**

You see, twice the triangle of 14 gives the triangle of 20. This means that the square of 14 + 1 + 20, that is 35, is also the triangle of 35 + 14, that is 49.

COMP. My computer would have told you the same thing much faster, so what is your point?

MATH. My purpose was to demonstrate how mathematical intuition can, so to say, unfold itself along with perspicuous representations. It seems that this is what Poincaré meant when he attributed the intuitions of Sophus Lie to his ability to think in pictures. Perhaps you know the fragment in which he compared Lie with Sonja Kowalewski, whom he considered a *logicienne*, as if mathematicians who do not think in pictures have no intuitions.

COMP. You know that I always had difficulties with the notion of intuition. I understand now that you regard intuition as a process with at least two aspects, an intuitive inference and a promising conclusion. I accept the notion of promising conclusion, but am still in doubt about those magnificent intuitive inferences. I agree that mathematicians sometimes make good guesses, but I hesitate to use the term intuition for them. Moreover good guesses can occur on any occasion, and certainly not only in connection with pictures as Poincaré seems to have thought.

MATH. This is just what I wanted to discuss, because I used the two examples of squangles myself in order to see if they could bring me further. However, I made use of a perspicuous representation, albeit not a pictorial one. Look:

$$6^2 = \tfrac{1}{2}\cdot 8\cdot 9$$
$$35^2 = \tfrac{1}{2}\cdot 49\cdot 50$$

When I wrote this down some days ago, it occurred to me that both squares have come about through two other numbers with a difference of 1, a square and the double of a square, 9 and 8, the double of 4, 49 and 50, the double of 25. That I still speak of an intuitive inference is because it came suddenly and because it was accompanied by a conviction of correctness or Helmholtz's *Überzeugung der Gewissheit*. I was certain that it was more than a good guess: I had found the characteristic property of the equations. This discovery was followed by the promising conclusion that a table with squares and their doubles would lead me even faster to more squangles than the earlier table.

COMP. So you began to make the new table in order to find the next squangle. What a primitive procedure in this age of the computer!

MATH. Be quiet, I am not yet finished. But let me first write down the relevant values of s(n) and 2s(n) and mark the outcomes with a difference 1:

n	s(n)	2s(n)
0	0	0
1	1	2
2	4	8
3	9	18
4	16	32
5	25	50
6	36	72
7	49	98
8	64	128
9	81	162
10	100	200
11	121	242
12	144	288
13	169	338
14	196	392
15	225	450
16	256	512
17	289	578

COMP. So you finally found that the square of 12·17 is equal to the triangle of 289. It took a lot of time, but I agree that it would have cost you more time to find that both 204^2 and ½·288·289 have the same outcome, let me see, 41616, if you had calculated all squares and triangular numbers up to this number. But I understand that this is not all there is.

MATH. Indeed. For I got the idea of making a list of the pairs of those numbers that successively produced 2 and 1, 8 and 9, 50 and 49, and 288 and 289:

$$
\begin{gathered}
(0, 1)\\
(1, 1)\\
(2, 3)\\
(5, 7)\\
(12, 17)
\end{gathered}
$$

COMP. I see, the next pair is (29, 41) and this means that the square of 29·41 is also a triangular number. Interesting, but did you prove the Fibonacci-like rule that supports my uh... intuitive inference?

MATH. Of course I did, in spite of my conviction of correctness. I even found the rule that lies behind the pairs that gave the desired values of t(n) and 2t(n), but only after I had enlarged the list with the help of the pairs (12, 17) and (29, 41) from the other list:

$$
\begin{gathered}
(0, 0)\\
(2, 3)\\
(14, 20)\\
(84, 119)\\
(492, 696)
\end{gathered}
$$

I saw immediately that each second component is 1 more than the sum of the three preceding numbers: 3 = 0 + 0 + 2 + 1, 20 = 2 + 3 + 14 + 1, 119 = 14 + 20 + 84 + 1, and 696 = 84 + 119 + 492 + 1, but it took somewhat longer before I found the rule for the first components: 14 = 4·3 + 0 + 2, 84 = 4·20 + 2 + 2 and 492 = 4·119 + 14 + 2. The proof that these rules are correct was not very simple, so I will not bother you with it.

COMP. Why did you rack your brains about a proof when you had no doubt of the correctness of the conclusion?

MATH. Well, that is just what makes mathematics challenging. Finding a proof is like a treasure hunt. But I am still not finished, for after I had found the rules for the lists, I wondered whether it would not be possible to derive a rule for the sequence of squangles themselves from them. However, the algebraic approach did not give much hope, because of the complexity of the Fibonacci-like rule. But when I looked again at the last list but one, I discovered that there is already a connection between the x-coordinates alone: from the third x-coordinate each is a simple linear combination of the two preceding ones: $x_{n+2} = 2x_{n+1} + x_n$ and the same holds for the y-coordinates. But then I turned to the squangles:

$$0, 1, 6, 35, 204, 1189, \ldots$$

I imagined that here, from the third term on, each term might also be a simple combination of the two preceding ones. I was right! Do you see it too?

COMP. I am sorry but I already left my computer too long alone. Thank you for your lecture, and see you later. (*Leaves the classroom while Math begins to scrawl on a piece of paper, trying for a generalization. Ten minutes later Comp returns.*)

MATH. Hallo Comp, did you set your computer to work?

COMP. Yes, and how! I wrote a program that calculated the first twelve squangles. It also found the rule that governs it. I shall write it down in your notation:

$$t_{n+2} = 6t_{n+1} - t_n$$

MATH. And now you are certainly going to argue that my work was superfluous? But who invented not only the problem, but also the kinds of sequences and rules that brought the problem to a solution? I begin almost to think that you could not find the last rule yourself!

COMP. Goodbye Math! (*leaves the classroom with a shrug of the shoulders*)

Notes

1. The above dialogue is by no means meant as a criticism of the work of creative computer scientists. Their approaches can be very original as I learned from Marjan Dragt, who wrote the program alluded to in the text.
2. The notion of 'perspicuous representation' played an important role in Wittgenstein's later work. Hubert Dreyfus (1967: 40) used it in his criticism of Artificial Intelligence research.
3. The notion of 'scheme' originally comes from Kant (1781: A 140), who distinguished 'schemes' from 'pictures'. An example of the latter is the following representation of the number five by five points: ·····, whereas a scheme would enable us to think of a number 'in general'. It would be rather a 'method for representing a number by a picture'

than the picture itself. It is conceivable that Kant's difficulties with this notion would have been less great, if he had made use of the notion of 'variable' or if he had acknowledged that mathematicians often 'draw figures' of all kinds. For the role of figures in the solution of problems, I refer to Polya's (1945) little book on invention and discovery.

4. 'Intuition' is, of course, a delicate subject. Scientists such as Helmholtz (1903), Poincaré (1905), and Mach (1905) acknowledged it as an important source of discoveries, and they already tried to get a grip of the relevant phenomena. Bertrand Russell (1948: 421-422) once gave the example of the sum of the first two, three and four cubic numbers, but he admitted that he did not know 'how to make explicit what guides mathematical intuition in such cases'.
5. The idea of presenting arguments in the form of a dialogue is nothing new. My first serious contact with this approach was through Heyting's (1958) book on intuitionism.

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