SMALLEST NUMBERS

(*Log comes to visit Math*.)

LOG. Still working on a pet problem, Math?

MATH. I am looking for the smallest numbers with exactly k representations as a sum of successive positive integers. This is what I have already found:

(k = 1)

3 is the smallest number with exactly one representation as a sum of successive positive integers:

3 = 1 + 2

(k = 2)

9 is the smallest number with exactly two representations as a sum of successive positive integers:

$$9 = 3 \cdot 3$$

9 = 4 + 5
= 2 + **3** + 4

(k = 3)

15 is the smallest number with exactly three representations as a sum of successive positive integers:

$$15 = 3 \cdot 5$$

15 = 7 + 8
= 4 + **5** + 6
= 1 + 2 + **3** + 4 + 5

(k = 5)

45 already has exactly five representations as a sum of successive positive integers:

$$45 = 3 \cdot 3 \cdot 5$$

45 = 22 + 23
= 14 + **15** + 16
= 5 + 6 + **7 + 8** + 9 + 10
= 7 + 8 + **9** + 10 + 11
= 1 + 2 + 3 + 4 + **5** + 6 + 7 + 8 + 9

The fact that 45 is a triangular number gives it a 'bonus'. This is not the case with 75 – also a product of factors 3 and 5. Nevertheless 75 has also exactly five representations as a sum of successive positive integers:

$$75 = 3 \cdot 5 \cdot 5$$

75 = 37 + 38
= 24 + **25** + 26
= 10 + 11 + **12 + 13** + 14 + 15
= 13 + 14 + **15** + 16 + 17
= 3 + 4+ 5 + 6 + **7 + 8** + 9 +10 + 11 + 12

63 is a smaller product of three odd prime numbers than 75:

$$63 = 3 \cdot 3 \cdot 7$$

However, 63 has exactly five representations as a sum of successive positive integers too:

63 = 31 + 32
= 20 + **21** + 22
= 8 + 9 + **10 + 11**+ 12 + 13
= 6 + 7 + 8 + **9** + 10 + 11 + 12
= 3 + 4 + 5 + 6 + **7** + 8 + 9 + 10 + 11

You see, I had serious problems with the value 4 for *k*. And when I investigated 105 – the product of the first three odd prime numbers – I found exactly seven representations as a sum of successive positive integers.

($k = 7$)

$$105 = 3 \cdot 5 \cdot 7$$

105 = 52 + 53
= 34 + **35** + 36
= 15 + 16 + **17 + 18** + 19 + 20
= 19 + 20 + **21** + 22 + 23
= 6 + 7 + 8 + 9 + **10 + 11** + 12 + 13 + 14 +1 5
= 12 + 13 + 14 + **15** + 16 + 17 + 18
= 1 + 2 + 3 + 4 + 5 + 6 + **7 + 8** + 9 + 10 + 11 + 12 + 13 + 14

LOG. Returning to the value 4, why not asking Comp for help?

(*Math sends an email message to Comp, explaining the problem. Meanwhile, Math and Log turn to the geometrical problem that Lyber Katz once mentioned to Martin Gardner: 'Three squares'.*)

COMP. (*By email, somewhat later.*) I am very busy with another computational task, but I could not resist the temptation of writing a little program for your problem. It gave solutions up to 35, but several outcomes are missing, for example for 12. This means that these values are greater than 20100, or do not exist at all.

MATH. (*Looking at Comp's results.)* The strange thing is that the smallest number with exactly four representations as a sum of successive positive integers is 81:

($k = 4$)

$$81 = 3 \cdot 3 \cdot 3 \cdot 3$$

81 = 40 + 41

= 26 + **27** + 28
= 11 + 12 + **13 + 14** + 15 + 16
= 5 + 6+ 7 + 8 + **9** + 10 + 11 + 12 + 13

LOG. And what did Comp find for $k = 6$?

MATH. Very interesting! It is again an even power of 3.

($k = 6$)

$$729 = 3·3·3·3·3·3$$

729 = 364 + 365
= 242 + **243** + 244
= 119 + 120 + **121 + 122** + 123 + 124
= 77 + 78 + 79 + 80 + **81** + 82 + 83 + 84 + 85
= 32 + 33 + ... + 39 + **40 + 41** + 42 + ... + 48 + 49
= 14 + 15 + ... + 26 + **27** + 28 + ... + 39 + 40

LOG. Until now, three even powers of 3 gave a minimum. I wonder if there is a logic behind it. The number of sums corresponds with the exponents. How many sums will 3^8 have? Eight?

MATH. $3^8 = 6561$ does not occur in Comp's list. In any case it is not a minimum with eight sums, because according to Comp that minimum is 225:

$$225 = 3·3·5·5$$

LOG. Let me see. The sums are reached obtained in the same way as in the case of 3^6. I will give only the middle terms:

$3^8 = ½(3^8 – 1) + ½(3^8 + 1)$
$= ... + 3^7 + ...$
$= ... + ½(3^7 – 1) + ½(3^7 + 1) + ...$
$= ... + 3^6 + ...$
$= ... + ½(3^6 – 1) + ½(3^6 + 1) + ...$
$= ... + 3^5 + ...$

$= \ldots + ½(3^5 – 1) + ½(3^5 + 1) + \ldots$
$= \ldots + 3^4 + \ldots$

MATH. Fine! Curiously, the smallest number with ten sums is missing in Comp's list. Now we have at least one number with ten sums, 3^{10}. Not only that, but also a number with twelve sums, 3^{12}. Still the smallest number with exactly fourteen representations as a sum of successive positive integers is already 2025, according to Comp. Isn't it amazing how capricious the development is!

LOG. Can you list all Comp's outcomes?

MATH. With pleasure:

$m(1) = 3$
$m(2) = 3^2$
$m(3) = 3·5$
$m(4) = 3^4$
$m(5) = 3^2·5$
$m(6) = 3^6$
$m(7) = 3·5·7$
$m(8) = 3^2·5^2$
$m(9) = 3^4·5$

$m(11) = 3^2·5·7$

$m(13) = 3^6·5$
$m(14) = 3^4·5^2$
$m(15) = 3^3·5·7$

$m(17) = 3^2·5^2·7$
$m(19) = 3^4·5·7$
$m(20) = 3^6·5·7$

$m(23) = 3^2·5·7·11$

$m(26) = 3^2·5^2·7^2$

$m(29) = 3^4 \cdot 5^2 \cdot 7$

$m(31) = 3^3 \cdot 5 \cdot 7 \cdot 11$

$m(35) = 3^2 \cdot 5^2 \cdot 7 \cdot 11$

LOG. I see some regularities, but they are not persistent. For example:

$m(7) = 3 \cdot 5 \cdot 7$
$m(11) = 3^2 \cdot 5 \cdot 7$
$m(15) = 3^3 \cdot 5 \cdot 7$
$m(19) = 3^4 \cdot 5 \cdot 7$
$m(23) = 3^2 \cdot 5 \cdot 7 \cdot 11$ instead of $3^5 \cdot 5 \cdot 7$.

MATH. Apparently, $3^2 \cdot 5 \cdot 7 \cdot 11$ is smaller than $3^5 \cdot 5 \cdot 7$. Yet adding a factor 3 brings about that the number of sums is enlarged by four.

LOG. That is your intuition. Nevertheless I will transcribe the outcomes of m(7) and m(11) in terms of the middle terms:

$3 \cdot 5 \cdot 7 = \frac{1}{2}(3 \cdot 5 \cdot 7 - 1) + \frac{1}{2}(3 \cdot 5 \cdot 7 + 1)$
$= \ldots + 5 \cdot 7 + \ldots$
$= \ldots + \frac{1}{2}(5 \cdot 7 - 1) + \frac{1}{2}(5 \cdot 7 + 1) + \ldots$
$= \ldots + 3 \cdot 7 + \ldots$
$= \ldots + \frac{1}{2}(3 \cdot 7 - 1) + \frac{1}{2}(3 \cdot 7 + 1) + \ldots$
$= \ldots + 3 \cdot 5 + \ldots$
$= \ldots + \frac{1}{2}(3 \cdot 5 - 1) + \frac{1}{2}(3 \cdot 5 + 1) + \ldots$

$3^2 \cdot 5 \cdot 7 = \frac{1}{2}(3^2 \cdot 5 \cdot 7 - 1) + \frac{1}{2}(3^2 \cdot 5 \cdot 7 + 1)$
$= \ldots + 3 \cdot 5 \cdot 7 + \ldots$
$= \ldots + \frac{1}{2}(3 \cdot 5 \cdot 7 - 1) + \frac{1}{2}(3 \cdot 5 \cdot 7 + 1) + \ldots$
$= \ldots + 3^2 \cdot 7 + \ldots$
$= \ldots + \frac{1}{2}(3^2 \cdot 7 - 1) + \frac{1}{2}(3^2 \cdot 7 + 1) + \ldots$
$= \ldots + 5 \cdot 7 + \ldots$
$= \ldots + \frac{1}{2}(5 \cdot 7 - 1) + \frac{1}{2}(5 \cdot 7 + 1) + \ldots$
$= \ldots + 3 \cdot 7 + \ldots$
$= \ldots + \frac{1}{2}(3 \cdot 7 - 1) + \frac{1}{2}(3 \cdot 7 + 1) + \ldots$
$= \ldots + 3 \cdot 5 + \ldots$

$= \ldots + \frac{1}{2}(3 \cdot 5 - 1) + \frac{1}{2}(3 \cdot 5 + 1) + \ldots$

MATH. Now we can see more clearly how the smallest numbers come about. Take, for example, $m(23) = 3^2 \cdot 5 \cdot 7 \cdot 11$. This number can be derived from $m(11) = 3^2 \cdot 5 \cdot 7$. Adding the factor 11 leads up to twelve more sums.

LOG. There are indeed six products containing the new factor 11. Each of them produces two new sums. I think that this approach gives us sufficient knowledge about the progression of the minima.

MATH. The practice remains more laborious. One must still choose the smallest number among the different outcomes of the applications of this procedure ...

LOG. Fortunately it was only a pet problem!

(*They part laughing*.)

Postscript. I am very much indebted to Dr. Jeroen Donkers, who wrote the computer program indispensable for this paper.