REDESCRIPTIONS AND SUBSTITUTIONS

MATH. Good morning, Comp, you look puzzled, what is wrong with you?

COMP. I am fine, but I spoke with one of your students, who follows your courses on the Philosophy of Artificial Intelligence. He told me that you made fun of the fact that some AI-researchers admired a computer proof of the first part of Euclid's Proposition 5 (*In isosceles triangles the angles at the base are equal to one another*). They were surprised that it needed no auxiliary lines, because it declared the isosceles triangle *ABC* such that *AC* and *BC* are "equal", congruent with the triangle *BAC*. (*Comp draws Figure 1 on Math's blackboard.*) I must say that the difference with Euclid's proof is indeed tremendous, if only we look at the figures. (*He also draws Figure 2.*)

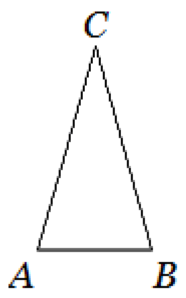

Figure 1

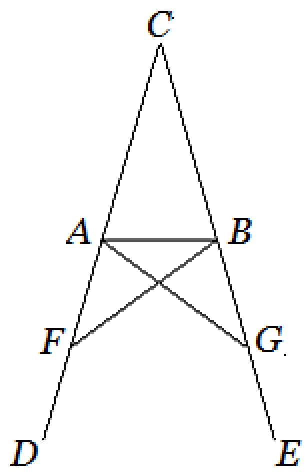

Figure 2

MATH. The question was that they thought that the proof was new, and that therefore computers could be creative. However, the proof was only new for themselves, because it was already given by Pappos at the end of the third century BC. Nevertheless we can draw some lessons out of this example.

First, it illustrates that a certain geometrical problem may be solved in different ways, which may be either complicated – Euclid's proof of Proposition 5 with the help of auxiliary lines (*Math points at Fig. 2*) – or simple – Pappos' proof without auxiliary lines. Second, the example suggests that different solutions may be the result of different procedures – the computer program had no procedure for finding a proof in the style of Euclid.

COMP. So you believe that things would have been different if the program had given Euclid's proof!

MATH. Yes, I do. But I also believe that Euclid certainly already knew the simple proof, but deliberately bypassed it, because he would then have missed the second part of Proposition 5, viz. that the angles under the angles – the angles *BAD* and *ABE* in the second figure – are equal to one another. He needed this conclusion in his proof of Proposition 6. This brings me to the third lesson from the example: different procedures may be based on different background knowledge or different purposes. Euclid's proof was not just an isolated exercise, but it formed part of an attempt to build a firm foundation for certain central propositions.

COMP. Can you imagine that something like that could also be done by a computer program?

MATH. Well, the problem is that the principles and procedures underlying Euclid's system are not well known. But how do we get more insight into these principles and procedures?

COMP. I see. This was the starting point of your research on productive problem solving. I read your discussion paper 'Aristarchus' visit to Euclid' and remember that it contains an analysis of Euclid's proof of the first twenty-six theorems (Propositions) of Book One of *The Elements*.

LOG. (*She enters Math's room and points at the figures on the blackboard.)* Hallo, boys, do you suffer from such a regression that you are discussing elementary geometry?

COMP. Did you not read Math's paper on Book One of Euclid's Elements? That is not a regression, but a reconstruction of the way in which Euclid may have build his system.

LOG. I am sorry, I have more interests in 'higher geometry', so to say. Now I am studying Lie transformations. Nevertheless I am anxious to hear what your findings were, Math.

MATH. Let me outline the main features with the help of an analysis of parts of some proofs. Even this restricted approach will give rise to some remarkable conclusions. Yet before doing this, I want to make some remarks on Euclid's hypothetical research program.

There is no reason to think that Euclid began by formulating axioms, and subsequently gave proofs of more or less important theorems. Instead, his problem may have been: to give proofs for what he considered to be central theorems of geometry. In the course of his attempts he may have encountered other, more basic theorems, which had to be proved, until he reached his primitive propositions, and primitive notions. In other words, Euclid would have had a combined retroductive and deductive approach – retroduction as the converse of deduction, that is, as the task of finding suitable premises for given conclusions, a kind of backwards search from certain central propositions. But what are his central propositions? On this point, we can construct deductive trees, or, working backwards, retroductive trees, starting from several hypothetical central propositions, such as Proposition 20 (*In any triangle two sides taken together in any manner are greater than the remaining one*), the later famous *triangle inequality*. I shall not attempt this here, but make a partial use of this strategy, by first taking Proposition 18 into consideration, notwithstanding the fact that Euclid's proof of this theorem presupposes that another, relatively important proposition has already been proved at an earlier stage.

In the translation by Heath (1978), Euclid's proof of Proposition 18 (*In any triangle the greater side subtends the greater angle*), is as follows:

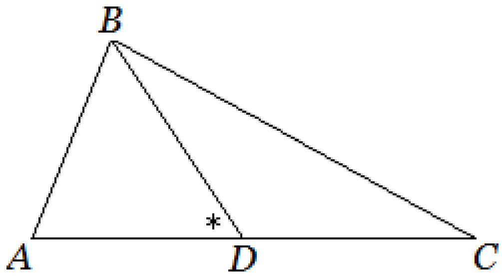

Figure 3

> For let *ABC* be a triangle having the side *AC* greater than *AB*; I say that the angle *ABC* is also greater than the angle *BCA*. For, since *AC* is greater than *AB*, let *AD be made equal to* AB [I. 3], and let *BD* be joined (Fig. 4). Then, since the angle *ADB* is an exterior angle of the triangle *BCD*, it is greater than the interior and opposite angle *DCB*. [I. 16]. [It is here that Euclid uses an earlier proved theorem, Proposition 16 (*In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles*). Euclid's proof is ingenious, though it is defective according to modern standards.] But the angle *ADB* is equal to the angle *ABD*, since the side *AB* is equal to *AD*; therefore the angle *ABD* is also greater than the angle *ACB*; therefore the angle *ABC* is much greater than the angle *ACB*. Therefore etc. Q. E. D.

Apparently we have here an example of how a certain element of the solution, in this case an angle, is described in two different ways. The angle *ADB*, which is first described as an exterior angle of a certain triangle, gets a redescription as an (interior) angle of a different triangle. In the terminology of Frege (1892), its *Art des Gegebenseins* – the manner in which the angles are presented – changes in the course of the proof. Visually, there is a remarkable agreement with Jastrow's famous rabbit-duck picture. The angle *ADB* can also be *seen* in two different ways, either as an exterior angle, or as an interior angle. Even the angle *ABD* is also perceived in two different ways, first as an interior angle of the triangle *ABD*, and then as a part of the (interior) angle *ABC* of the triangle *ABC*. Moreover, the line *AB* is first described as a side of the triangle *ABC*, and then as a side of the triangle *ABD*. These and many more similar examples led me to speak of a special procedure called redescription.

LOG. I know almost nothing about philosophy, but did Wittgenstein not use the rabbit-duck picture in his *Philosophical Investigations* in order to distinguish different uses of the verb 'to see'?

MATH. Moreover he gave also other, more abstract examples, and hinted at their significance for problem solving when he pointed out that differemt sides of a triangle can either be seen as apex or as base, dependent on the applications which one wants to make of the figure. He even called the substratum of such experiences the mastery of a technique.

When I used the expression 'seeing as', this was also not meant as a technical term, but only as an aid to better understanding. One must be very careful not to go beyond the data as long as they are analyzed. Proofs of geometrical theorems are given in the form of statements. In other tasks, solutions may consist only of certain non-linguistic actions. In that case, there are no descriptions at all, and then the analyst is tempted to use such expressions as 'seeing as' in order to characterize certain steps of a particular solution. Match-stick problems present clear examples. Perhaps we can discuss them later.

LOG. There is also the problem, already touched upon by the early Wittgenstein (1921), whether we 'really see two different facts' in those cases in which a figure can be seen in two ways. What do you think about it?

MATH. It seems safer to say that a redescription corresponds, in a sense, with a different manner of seeing. This is in accordance with William

James's (1890) description of what happens in so-called cases of ambiguity of perception.

COMP. I thought that we were discussing AI research, but now you are both talking about psychology or even about philosophy with which I want nothing to do.

MATH. OK. So I ask you: how would a computer program solve the problem of proving Proposition 18, with all foregoing theorems at its disposal?

COMP. It seems that it would need a counterpart of the procedure of redescription in the form of explicit rules, if only for redescriptions of angles.

MATH. Yes, and it is possible to investigate the corresponding implementation problem. but I shall not go further into this matter. I only point out that the angle *ABD* in Euclid's proof does not change its name, as is the case with the triangle in Pappos' proof. Of course human beings have the advantage above computers that they can inspect the figures, in their attempts to solve a geometrical problem.)

LOG. I understand that your analysis of Euclid's proofs demonstrates this?

MATH. Sure, so let me continue my story by turning to Euclid's proof of Proposition 20, the *triangular inequality*. Again, Euclid's proof presupposes another theorem, namely Proposition 19 (*In any triangle the greater angle is subtended by the greater side*), which is not geometrically proved by Euclid, but derived as a logical conclusion from Proposition 5 and Proposition 18. Here follows the proof of Proposition 20, again in the translation by Heath (1978):

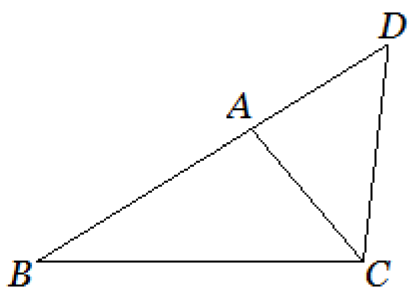

Figure 4

> For let *ABC* be a triangle; I say that in the triangle ABC two sides taken together in any manner are greater than the remaining one, namely
>
> *BA*, *AC* greater than *BC*,
> *AB*, *BC* greater than *AC*,
> *BC*, *CA* greater than *AB*.
>
> For let *BA* be drawn through the point *D*, let *DA* be made equal to *CA*, and let *DC* be joined (Fig. 5). Then, since *DA* is equal to *AC*, the angle *ADC* is also equal to the angle *ACD*; [I. 5] therefore the angle *BCD* is greater than the angle *ADC*. [*C.N. 5*] And, since *DCB* is a triangle having the angle *BCD* greater than the angle *BDC*, and the greater angle is subtended by the greater side, [I. 19] therefore *DB* is greater than *BC*. Similarly we can prove that *AB*, BC are also greater than *CA*, and *BC*, *CA* than *AB*. Therefore etc.
> Q. E. D.

Euclid's solution of the problem of comparing the sum of the lengths of two sides of a triangle with the other side was as follows. He replaced one of the former sides by another line, in such a way that one line is generated with a length equal to the sum of the lengths of the first two sides. This analysis brought me to the introduction of a technical term: substitution. In the above example, Euclid had to construct a new line, in order to make a substitution possible, and it is clear that this required – what some people would call – an act of creation. It can further be noticed that Euclid's proof also contains a redescription: the angle *ACD* is first seen as a part of the triangle *ACD*, and then as a part of the angle *BCD*.

LOG. Apparently the occurrence of substitutions in elementary geometry is based on the presence of congruent figures. It follows that one must be careful to generalize the procedure of substitution to other domains!

MATH. Nevertheless I will remind you of an example of a common sense solution which is even based on two substitutions. I mean Duncker's (1938) soluution of his famous mountain trip problem:

> Assume that someone successively ascends and descends a mountain during the same hours on two days, can he be on (at least) one and the same spot at exactly the same time at each day, assuming that there is only one way to go?
>
> Replace one person by two persons, and two days by one day!

But let us stay for a moment with Euclid’s solutions, in order to see more clearly what substitutions may involve.

An analysis of Euclid’s solution for the proof of Proposition 16 (*In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles*) shows that much has to be done, before a suitable substitution can be carried out.

COMP. This is not unimportant with respect to AI research; in this case, the mechanization of geometrical thought processes. How did Euclid come to his solution?

LOG. You can only give a rational reconstruction of Euclid’s thought process, Math, and in such an approach you should use confirmed hypotheses about his way of thinking, which you do not have, have n’t you?

MATH. There is nothing controversial in attributing to Euclid a *means-ends analysis*. In order to make the exterior angle in question directly comparable to a certain interior and opposite angle, he needed a substitution of the latter angle by an angle that is a part of the former angle. The appropriate way of reaching this goal was, apparently, the construction of a congruent triangle. This line of thought enables us to understand why Euclid proceeded as he did, again following Heath (1978):

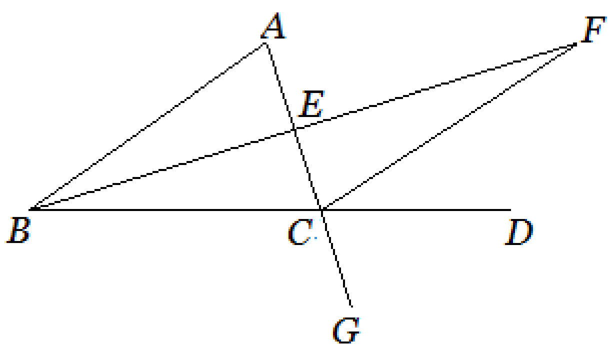

Figure 5

> Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*; I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA*, *BAC*. Let *AC* be bisected at *E* [I. 10], and let *BE* be joined and produced in a straight line to *F*. Let *EF* be made equal to *BE*, [I. 3] let *FC* be joined [Post. 1], and let *AC* be drawn thorough to *G*. [Post. 2] [etc. (Fig. 6)]

The *retroductive* scheme that underlies this proof can be formulated as follows:

$$x > y := x \supset z \wedge z \equiv y$$

This can be read as follows: x is greater than y becomes: x strictly contains z and z is congruent with y, for some z. This means that if we want to prove that x is greater than y, we may try to prove that x contains a z as a genuine part such that z is congruent with y.

LOG. Do you know that Euclid's proof is defective, as he did not prove that *FC* falls within the exterior angle *ACD*?

MATH. Yes, I do, but this is not a point with which I am here concerned. I see the proof as another datum for the identification of the procedure called substitution. It also reveals that elements which make a successful substitution possible do not necessarily have to be given, but may have to be produced, or created, by the problem solver. In traditional terms, the problem solver might use his imagination in order to achieve this, a fact that deserves closer examination. However, within the analysis of the data, there is no place for a discussion of such psychological questions. The same holds for the assertion that it is very easy to reproduce Euclid's proof of Proposition 16 by heart as soon as the above reconstruction of Euclid's means-ends analysis has been recalled. These subjects should be dealt with in connection with didactic applications.

COMP. You repeatedly remarked that your analysis is, in fact, a rational reconstruction. One step further, and a rational reconstruction of this rational reconstruction will have the form of a computer program! After all, psychological aspects are left out of consideration!

MATH. The difficulty remains that redescriptions may make use of descriptions of figures that are not present in the given figure. A remarkable example is a special solution of the famous square-with-inside-triangles problem (Figure 6):

> Given is a square *ABCD* and a point *E* inside this square such that the angles *EAB* and *EBA* are 15 degrees each; show that the triangle *CDE* is equilateral.
>
> The problem solver may make an appeal to a regular 12-angle (Figure 7), in order to draw the solution-producing auxiliary lines.

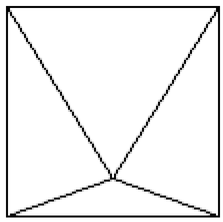

Figure 6

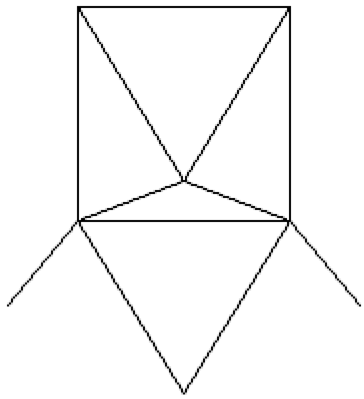

Figure 7

It is worth while to ask what a computer program would look like, if it would be able to give this solution. Even if its database would contain a description of a regular dodecagon which enabled it to conclude that *AE* and *EB* can be regarded as successive sides of such a figure, it should still have the heuristic possibility of construing the center *M* of the circumscribed circle.

LOG. Human beings must also have previous knowledge to produce this solution!

MATH. But they might imagine the potentially generated triangle *AMB* and *suppose* that it is congruent with the triangle *DEC*. The situation is comparable with Euclid' own proof of Proposition 5. This subject can best be dealt with in a discussion of 'intuition' (*Anschauung*), in connection with the aid which is given by the figure that is actually present in the problem situation. Perhaps we can talk about it another time?

LOG. I always feel uneasy when someone talks about intuitions, but I am looking forward to hear what you think about this subject. Maybe tomorrow?

MATH. With pleasure, but let me continue my outline of the procedures of redescription and substitution. It can be easily shown that they also play an important role in other proofs by Euclid. Therefore it can be asked which conditions make these procedures possible. Log mentioned congruence properties in the case of substitutions, but how can redescriptions come about? One trivial condition is that the required linguistic means must be available. For example, as long as the notion of an 'exterior angle' is not yet introduced, the corresponding redescription of an angle of a triangle cannot be given. However, it is conspicuous that Euclid gave no definition of exterior angles, but introduced the notion simultaneously with his formulation of Proposition 16. Other, simpler examples of redescriptions show that, already by itself, the notion of 'part of' gives occasion to redescriptions, as soon as a certain side, angle or even point can be regarded as forming part of two different triangles.

LOG. Returning to your first example, the different proofs of Proposition 5, I wonder whether the two different descriptions of a figure such as the isosceles triangle of Proposition 5 were already available from the beginning in Euclid's system. Pappos apparently allowed this possibility. But then one might have asked him the question: 'are *ABC* and *BAC two* triangles if *AC* is congruent with *BC*, or are '*ABC*' and '*BAC*' two different descriptions of one and the same triangle?' Of course it would be highly speculative to ask if Euclid deliberately avoided this 'philosophical' question when he gave his complicated proof of Proposition 5...

MATH. Allright. I will leave Euclid aside. You may know that the phenomenon of different descriptions of one and the same object is perhaps the most famous issue in modern analytical philosophy since Frege (1892) drew attention to it in his classical essay *Űber Sinn und Bedeutung*, curiously enough with the help of an example taken from elementary geometry. I quote from the English translation by Herbert Feigl (1949) (*He takes a book from his book case.)*:

> Let *a*, *b*, *c* be straight lines which connect the corners of a triangle with the midpoints of the opposite sides. The point of intersection of *a* and *b* is then the same as that of *b* and *c*. Thus we have different designations of the same point and these names ('intersection of *a* and *b*', 'intersection of *b* and *c*') indicate also the manner in which these points are presented.

Frege immediately made clear, after this example, how common this phenomenon is, when he gave the example of two descriptions, 'the evening star' and 'the morning star', for the same planet. Therefore it would not be at all strange if redescriptions would also occur in other domains than elementary geometry. The question is only whether they contribute essentially to the solution of problems outside geometry, so let us first have a look at elementary arithmetic.

Everyone knows the story of the young Gauss, who solved the problem of adding the natural numbers from 1 to 100 by transforming – at least in thought - the expression '1 + 2 + ... + 99 + 100' as '(1 + 100) + (2 + 99) + ... + (50 + 51)'. Wertheimer, who devoted a whole chapter of his book *Productive Thinking* to this story, considered the solution an example of ‘regrouping’, or, in general, of ‘reorganization’. These expressions are akin to Duncker’s notion of ‘restructuring’. However, my data, solutions of geometrical problems, are better described with the help of my notion of ‘redescription’. That a particular description can be *transformed* into another description is another question. Let us therefore leave Gauss in peace, and concentrate on other examples of elementary arithmetic, in which no transformation in the sense of Wertheimer takes place.

LOG. There is another story, this time about the famous Ramanujan, who – on his death-bed – answered the question posed by Hardy if there was anything remarkable about his taxi-cab number 1729 as follows: ‘It is the smallest natural number that can be written in two ways as the sum of two cubic numbers’. As a matter of fact, 1729 can indeed be described both as ‘1083 + 983’ and as ‘1283 + 183’; Ramanujan knew this, of course, and also that there is no smaller number than 1729 with this property. That is why he could give, in your terms, redescriptions – even four! – of a given number.

MATH. Very nice! This example reminds me of a discussion of problem solving by Jevons. Look! (*Math takes an edition of The Principles of Science from his book case.*) Jevons discussed the difficulty of performing inverse operations and then he remarked the following:

> Given any two numbers, we may by a simple and infallible process obtain their product; but when a large number is given it is quite another matter to determine its factors. Can the reader say what two numbers multiplied together will produce the number 8,616,460,799? I think it is unlikely that anyone but myself will ever know; for they are two large prime numbers, and can only be rediscovered by trying in succession a long series of prime divisors until the right one be fallen upon. The work would probably occupy a good computer for many weeks, but it did not occupy me many minutes to multiply the two factors together.

COMP. I did not know that there existed computers in Jevons’ time, although he invented and designed a logical machine, so what did he mean by computers?

MATH. Computers were in general retired army officers who worked for mathematicians and spent a lot of time in doing huge calculations. However, when a smaller number is given, Jevons’ problem may easily be solved with a redescription, on the condition that the problem solver has some elementary knowledge of algebra. Can you say what two numbers multiplied together will produce the number 1591? Yes, 1591 = 1600 – 9, or $40^2 - 3^2$, and, since $a^2 - b^2 = (a + b)(a - b)$, the desired numbers are 43 and 37. The example is in so far instructive that it shows how the possibility for redescriptions to solve a certain problem depends on the specific knowledge about the specific domain to which the problem belongs.

Jevons (1892) also gave an example of a problem that is familiar to every psychologist, the task of continuing a series of numbers. Talking about 'the inductive process', he wrote:

> To illustrate the passage from the known to the apparently unknown, let us suppose that the phenomena under investigation consist of numbers, and that the following six numbers being exhibited to us, we are required to infer the character of the next in the series:
> 5, 15, 35, 45, 65, 95.
> The question first of all arises, How may we describe this series of numbers?

LOG. It is possible to continue the series in the same way as it began, as 5, 15, 35, 45, 65, 95, 5, 15, 35, 45, 65, 95, or simply to go backwards in order to get 5, 15, 35, 45, 65, 95, 65, 45, 35, 15, 5, or whatever you want. In fact, such tasks are silly, notwithstanding the fact that psychologists still believe in their significance. But I understand that this not what Jevons meant?

MATH: What he meant, was a redescription of each term of the series as '2 less than a prime number ending – in the decimal notation – on a 7', so that the series can be redescribed as: 7 – 2, 17 – 2, 37 – 2, 47 – 2, 67 – 2, 97 – 2, and then the next term is 107 – 2, or 105. We can imagine that this problem is remolded in the form of a question-and-answer game, but that is not my concern here. What matters is that arithmetical problems may have solutions in which redescriptions are used because of the unlimited possibilities of redescribing numbers.

COMP. Am I right as I conclude that even a beginner in arithmetic may give a solution for an addition such as 85 – 19, by first calculating 85 – 15 and then 70 – 4?

MATH. That is right, and I thnk that it would be good if teachers explicitly instructed students to look for significant redescriptions. Of course such teachings should be based on a methodologically sound approach, which would require a prior delineation of the data, for example the problem solutions in a text-book for beginners. The interesting question is not whether redescriptions occur in elementary arithmetic, for there is an abundance of evidence that they do occur, but to what extent traditional introductions to arithmetic exploit the possibilities of redescription more or less systematically. Do they take into account that the ability to solve arithmetical problems increases with the insight into these possibilities? Are there verbal means for describing special substitutional procedures and, if so, should the students be informed about them? Such questions are best dealt with in connection with an examination of the (didactic) purposes of text-books.

LOG. It is possible that this will lead to a greater mastery of arithemetical technique in the sense of Wittgenstein, but I want to see it first.

MATH. My idea is that students must indeed be informed about higher points of view, also with mathematics, in other words, get lessons in the grammar of mathematics, just as they learn about the grammar of languages.

Speaking of languages, redescriptions occur also in every day reasonings.

I confine myself to one example. It is derived from an essay by Peirce (1878), in his *Deduction, induction, and hypothesis*. I have a copy at hand.

His main subject in this essay is hypothesis formation, later called abduction, which was considered by him as one of the two inversions of a classical deductive syllogism. But for my purpose it is important that Peirce's paradigmatic example of an hypothetical solution of an every-day problem contains a redescription in non-technical language:

> I once landed at a seaport in a Turkish province; and, as I was walking up to the house which I was to visit, I met a man upon horseback, surrounded by four horsemen holding a canopy over his head. As the governor of the province was the only personage I could think of who would be so greatly honored, I inferred that this was he. This was an hypothesis.

It follows that redescriptions are not restricted to mathematical problem solving.

COMP. I remember an example of a silent reading test, in which the solution to the main problem is the insight that the beautifulcow which a farmer later on a day buys, is the same cow which was missing after the depart early in the morning of a foreign guest whom he had offered a bed in the preceding night. But there is, of course, no theory behind such examples.

MATH. That is different in mathematics. We have seen that a possibility of the other procedure, substitution, in geometry depended on congruence properties: congruent sides have the same length, for example. And there are more circumstances that make substitution possible, such as the fact that different figures may have the same area; Euclid uses this in his proof of the Pythagorean theorem, an astonishing demonstration of Euclid's virtuosity in the use of this procedure. Another opportunity for substitution is shown in an example given by Duncker (1935): the problem of proving that the sum of the distances to the foci of an ellipse is greater for a point outside the ellipse than for a point on the ellipse (Figure 8).

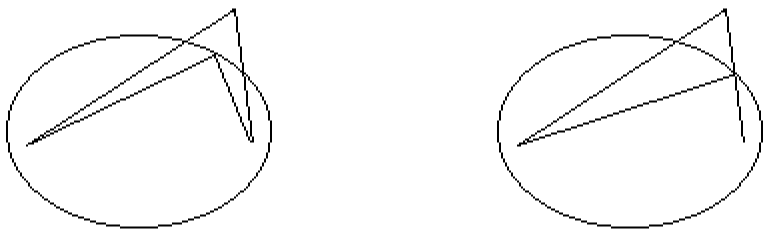

Figure 8

Duncker emphasized the difficulty of such procedures for some students; he stressed that there are people who find such restructurings and shifts of function within a system difficult, but also those who find them easy. Finding a significant substitution may appear to a student of the first category as an important discovery indeed, but this does not exclude the possibility that the art of proving in elementary geometry can to a certain degree be learned. Explicit formulations of specific substitutional (and redescriptive) procedures may contribute to this aim.

The fact that, even in geometry, suitable conditions make suitable substitutions possible, indicates that there is a more general condition: the presence of a certain invariance property. In Duncker's ellipse example it is the definition of the ellipse that already embodies such a property, and one would almost conclude that the success of elementary geometry, that is the abundance of provable theorems, is due to its man-made character. But fortunately, *invariances* are not restricted to mathematics. The natural sciences owe their existence to them, so to say. Therefore it does not come as a surprise that the hypothesis that substitutions play a role in physics is confirmed. We only need to examine Archimedes' well-known proof of the law of the lever in order to find a fine example of substitution. Similar confirmations can be found in other proofs in mechanics, by Stevin and Huygens, as we can see from Mach's book on the history of mechanics. But perhaps the father of the modern scientific method, Galileo, gave the most spectacular example of a successful substitution.

Instead of experiments with free fall, Galileo (1638/1952) took recourse to constructions of slightly inclined planes. It enabled him to perform time measurements, using water streaming out of a reservoir. Though this can already be seen as an example of substitution, it is the substitution of the free fall by a movement along the inclined plane that is perhaps Galileo's most brilliant idea. For he justified this substitution with an appeal to an *explicit assumption*, namely that *the speeds acquired by one and the same body moving down planes of different inclinations are equal when the heights of these planes are equal*. In other words, Galileo *postulated* invariance, and, by a miracle, he was successful!

It looks like an over-bold hypothesis, and I remember that Mach called it an assumption that appears to us as somewhat risky, although Galileo had Sagredi say that it appeared to him so reasonable that it might be conceded without question. More precise (*Math takes again a book*):

> All resistance and opposition having been removed, my reason tells me at once that a heavy and perfectly round ball descending along the lines *CA*, *CD*, *CB* would reach the terminal points *A*, *D*, *B*, with equal momentums.

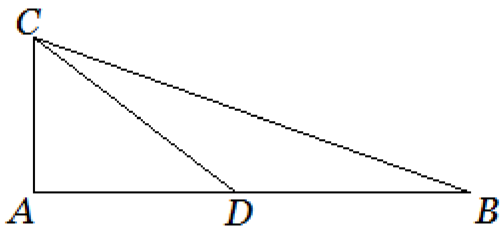

Figure 9

Was Galileo joking? Listen to Salviati: 'Your words are very plausible; but I hope by experiment to increase the possibility to an extent which shall be little short of a demonstration.'. It is followed by Galileo's famous pendulum proof, a combination of amazing redescriptions and substitutions. Of course, Galileo's applications of these procedures serve a different purpose than those of Euclid, and besides the question whether a particular substitution is correct depends on the domain in which it is applied. But I think that the occurrence of substitutions outside geometry has been made sufficiently clear by the given examples.

LOG. To me, the applications in geometry are sufficient. I will see if I can find them in the geometry of differential equations. Nice talking to you. Bye. (*She leaves the room.*)

COMP. Do you think that the insight into redesciptions and substitutions wiil help in writing artificial intelligent computer programs?

MATH. I am not much interested in AI programs as such, but it is possible that we learn more about problem solving procedures in attempts to write programs with which serious problems are solved.

COMP. I will give it a try and inform you about my results. Thank you, Math. (*He raises his hand and goes to his own room.*)

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