RAMANUJANIANA

(*It is the last working day of the year. Math, Comp, and Log are drinking coffee in the canteen of the L. E. J. Brouwer Institute, when Comp's attention is drawn by an interview in a newspaper in which the well-known anecdote of Ramanujan is mentioned. Ramanujan called the number of Hardy's taxicab, 1729, extraordinary, because it is the smallest number that can be twice written as the sum of two cubic numbers.*)

COMP. You are also a number freak, Math. Do you also have such interesting insights?

MATH. I'm not so strongly impressed about Ramanujan's remark to Hardy as most people. After all, someone who knows the first fifteen cubic numbers by heart, as Ramanujan undoubtedly did, can't have failed to see that 1000 + 729 is just 1 more than 1728. Nevertheless, I asked myself if I could invent a similar, but less easily solvable problem. My principle of variation helped me, as you understand. To present it in such a way that the variation comes out clearly: instead of two sums of powers of degree three, I took three sums of powers of degree two.

COMP. So you asked for the smallest number that can be written three times as the sum of two squares?

MATH. Right, and because I know the first forty squares by heart, I trusted that I could solve this problem.

LOG. But how could you be sure that there are any solutions at all?

COMP. What? That's not a problem to me; I know the first four irreducible Pythagorean triples by heart, and already the first three give a solution.

LOG. Show it!

COMP. We have $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, and $7^2 + 24^2 = 25^2$, and the least common multiple of 5, 13, and 25 gives a solution: 325^2. I will write it down for you (*he takes a paper napkin*):

$$(5\cdot13\cdot\underline{3})^2 + (5\cdot13\cdot\underline{4})^2 = (5\cdot13\cdot\underline{5})^2$$
$$(5\cdot5\cdot\underline{5})^2 + (5\cdot5\cdot\underline{12})^2 = (5\cdot5\cdot\underline{13})^2$$
$$(13\cdot\underline{7})^2 + (13\cdot\underline{24})^2 = (13\cdot\underline{25})^2$$

LOG. I am convinced!

MATH. 325^2 is surely not the smallest number with the desired property, but nevertheless it is very special. Comp, I thank you very much for this outcome, because the smallest number that is three times the sum of two squares is, believe it or not, 325!

LOG. I don't believe in mathematical magic, but I admit that it is remarkable!

COMP. Please give the three sums, Math.

MATH. 325 = 324 + 1, 325 = 289 + 36, and 325 = 225 + 100

LOG. How did you find it?

MATH. The first sum I tried was 365, just for fun, because the year is almost over, and I saw or knew that it is both the sum of 169 and 196, and the sum of 361 and 4, but there was no more to it. Then I tried 325, because it is 324 + 1, similar to Ramanujan's 1728 + 1, and I found the three sums. Thereafter I tried numbers smaller than 325, but this yielded at most only numbers with two sums.

COMP. I will not underestimate your arithmetical capacities, Math, but still I will check your conclusion on my computer if you don't mind ...

MATH. Good to hear that you keep working in the vacation period! I also have problems that bother me, so have a good time and see you next year!

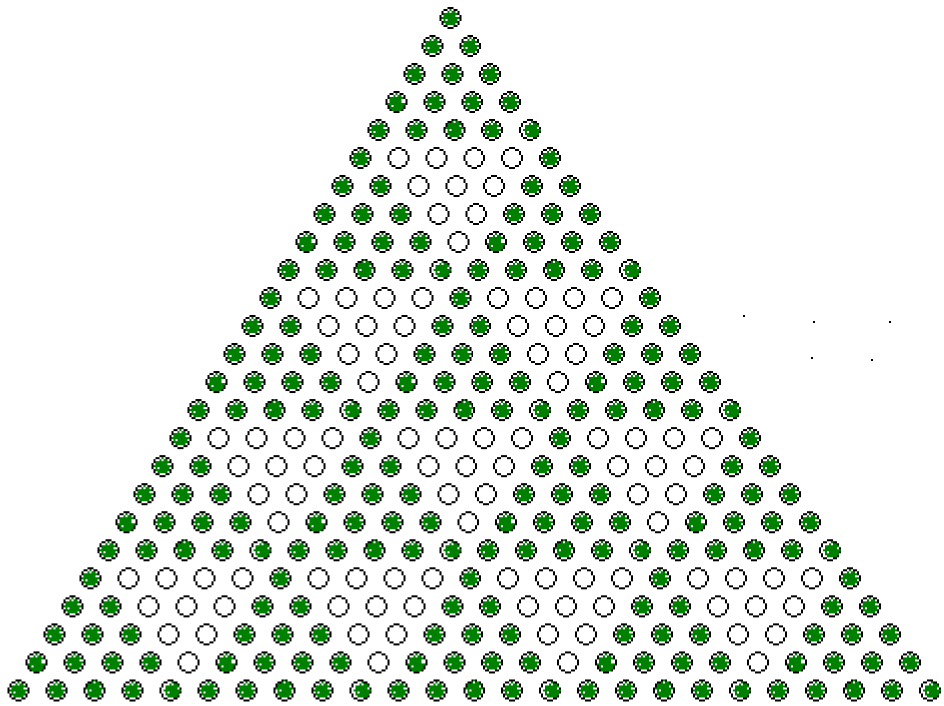
LOG. Are you perhaps trying to invent more Ramanujanean problems, Math, or is there still more to tell about 325?

MATH. Yes and yes, and I can already answer your second question: 325 is the triangular of a square number, so it can be written as the sum of the squares of two successive triangular numbers, just as the famous or notorious number 666.

LOG. I see (*she writes on the napkin*):

$$325 = 15^2 + 10^2, \text{ or } t(s(5)) = s(t(5)) + s(t(4))$$

MATH. This can be represented in a perspicuous figure, as you both know. It becomes my season's greetings picture (*he takes a card out of his pocket*):



Best wishes and a happy new year!

(*They all go their own way.*)

(*When Math is still in his room, trying to bring some order in his papers before leaving the Institute, Comp enters.*)

COMP. I checked your solution, and you were right, 325 is the smallest number with the desired property! But I did more, I looked for the smallest number that is three times the sum of three squares. It appeared to be 101, because 101 = 1 + 36 + 64, 4 + 16 + 81, and 16 + 36 + 49. However, there are so many numbers which are the sum of three squares, 110, 126, 134, 146, and so on, that I wondered what would be the use of this exercise.

MATH. Would it perhaps be better to follow my principle of variation, and look for the smallest number that is four times the sum of three squares?

COMP. I will do my best, please stay in your room for a while! (*He leaves, and returns half an hour later.*)

COMP. The smallest number that is four times the sum of three squares is 161, and the smallest number that is five times the sum of four squares is 126. I don't think it's worthwhile to pursue this further. Therefore I wrote a program in which the principle of generalization is applied to your original problem: to find the smallest numbers that are *n* times the sum of two squares. However, it isn't guaranteed that the outcomes will be optimal. I hereby give you the results in the form I got them. I am not sure about 11, 17, 19, and 21. Therefore I omitted them. The other missing values gave as yet no solutions. If there are such, they are too large for my program. Moreover it got stuck after 32.

***n* = 1**
5 = (s(1)+s(2))

***n* = 2**
65 = (s(1)+s(8)) = (s(4)+s(7))

***n* = 3**
325 = (s(1)+s(18)) = (s(6)+s(17)) = (s(10)+s(15))

***n* = 4**
1105 = (s(4)+s(33)) = (s(9)+s(32)) = (s(12)+s(31)) = (s(23)+s(24))

***n* = 5**
8125 = (s(5)+s(90)) = (s(27)+s(86)) = (s(30)+s(85)) = (s(50)+s(75)) = (s(58)+s(69))

***n* = 6**
5525 = (s(7)+s(74)) = (s(14)+s(73)) = (s(22)+s(71)) = (s(25)+s(70)) = (s(41)+s(62)) = (s(50)+s(55))

***n* = 7**
105625 = (s(36)+s(323)) = (s(80)+s(315)) = (s(91)+s(312)) = (s(125)+s(300)) = (s(165)+s(280)) = (s(195)+s(260)) = (s(204)+s(253))

***n* = 8**
27625 = (s(20)+s(165)) = (s(27)+s(164)) = (s(45)+s(160)) = (s(60)+s(155)) = (s(83)+s(144)) = (s(88)+s(141)) = (s(101)+s(132)) = (s(115)+s(120))

***n* = 9**
71825 = (s(1)+s(268)) = (s(40)+s(265)) = (s(65)+s(260)) = (s(76)+s(257)) = (s(104)+s(247)) = (s(127)+s(236)) = (s(160)+s(215)) = (s(169)+s(208)) = (s(188)+s(191))

***n* = 10**
138125 = (s(22)+s(371)) = (s(35)+s(370)) = (s(70)+s(365)) = (s(110)+s(355)) = (s(125)+s(350)) = (s(163)+s(334)) = (s(194)+s(317)) = (s(205)+s(310)) = (s(218)+s(301)) = (s(250)+s(275))

***n* = 12**
160225 = (s(15)+s(400)) = (s(32)+s(399)) = (s(76)+s(393)) = (s(81)+s(392)) = (s(113)+s(384)) = (s(140)+s(375)) = (s(175)+s(360)) = (s(183)+s(356)) = (s(216)+s(337)) = (s(228)+s(329)) = (s(252)+s(311)) = (s(265)+s(300))

***n* = 13**
1221025 = (s(47)+s(1104)) = (s(105)+s(1100)) = (s(169)+s(1092)) = (s(264)+s(1073)) = (s(272)+s(1071)) = (s(425)+s(1020)) = (s(468)+s(1001)) = (s(520)+s(975)) = (s(561)+s(952)) = (s(576)+s(943)) = (s(663)+s(884)) = (s(700)+s(855)) = (s(744)+s(817))

***n* = 14**

3453125 = (s(31)+s(1858)) = (s(110)+s(1855)) = (s(175)+s(1850)) = (s(350)+s(1825)) = (s(550)+s(1775)) = (s(625)+s(1750)) = (s(686)+s(1727)) = (s(815)+s(1670)) = (s(847)+s(1654)) = (s(970)+s(1585)) = (s(1025)+s(1550)) = (s(1090)+s(1505)) = (s(1201)+s(1418)) = (s(1250)+s(1375))

***n* = 15**
1795625 = (s(5)+s(1340)) = (s(52)+s(1339)) = (s(179)+s(1328)) = (s(200)+s(1325)) = (s(325)+s(1300)) = (s(380)+s(1285)) = (s(467)+s(1256)) = (s(520)+s(1235)) = (s(563)+s(1216)) = (s(635)+s(1180)) = (s(676)+s(1157)) = (s(800)+s(1075)) = (s(808)+s(1069)) = (s(845)+s(1040)) = (s(940)+s(955))

***n* = 16**
801125 = (s(10)+s(895)) = (s(95)+s(890)) = (s(127)+s(886)) = (s(158)+s(881)) = (s(193)+s(874)) = (s(230)+s(865)) = (s(241)+s(862)) = (s(335)+s(830)) = (s(370)+s(815)) = (s(430)+s(785)) = (s(458)+s(769)) = (s(463)+s(766)) = (s(529)+s(722)) = (s(545)+s(710)) = (s(554)+s(703)) = (s(610)+s(655))

***n* = 18**
2082925 = (s(26)+s(1443)) = (s(134)+s(1437)) = (s(163)+s(1434)) = (s(195)+s(1430)) = (s(330)+s(1405)) = (s(370)+s(1395)) = (s(429)+s(1378)) = (s(531)+s(1342)) = (s(541)+s(1338)) = (s(558)+s(1331)) = (s(579)+s(1322)) = (s(702)+s(1261)) = (s(730)+s(1245)) = (s(755)+s(1230)) = (s(845)+s(1170)) = (s(894)+s(1133)) = (s(926)+s(1107)) = (s(1014)+s(1027))

***n* = 20**
4005625 = (s(75)+s(2000)) = (s(147)+s(1996)) = (s(160)+s(1995)) = (s(336)+s(1973)) = (s(380)+s(1965)) = (s(405)+s(1960)) = (s(488)+s(1941)) = (s(565)+s(1920)) = (s(632)+s(1899)) = (s(700)+s(1875)) = (s(852)+s(1811)) = (s(875)+s(1800)) = (s(915)+s(1780)) = (s(1069)+s(1692)) = (s(1080)+s(1685)) = (s(1140)+s(1645)) = (s(1197)+ s(1604)) = (s(1260)+s(1555)) = (s(1325)+s(1500)) = (s(1344)+s(1483))

***n* = 22**
30525625 = (s(235)+s(5520)) = (s(525)+s(5500)) = (s(612)+s(5491)) = (s(845)+s(5460)) = (s(1036)+s(5427)) = (s(1131)+s(5408)) = (s(1320)+s(5365)) = (s(1360)+s(5355)) = (s(1547)+s(5304)) = (s(2044)+s(5133)) = (s(2125)+s(5100)) = (s(2163)+s(5084)) = (s(2340)+s(5005)) = (s(2600)+s(4875)) = (s(2805)+s(4760)) = (s(2880)+s(4715)) = (s(3124)+s(4557)) = (s(3315)+s(4420)) = (s(3468)+s(4301)) = (s(3500)+s(4275)) = (s(3720)+s(4085)) = (s(3861)+s(3952))

***n* = 24**
5928325 = (s(63)+s(2434)) = (s(94)+s(2433)) = (s(207)+s(2426)) = (s(294)+s(2417)) = (s(310)+s(2415)) = (s(465)+s(2390)) = (s(490)+s(2385)) = (s(591)+s(2362)) = (s(690)+s(2335)) = (s(742)+s(2319)) = (s(849)+s(2282)) = (s(878)+s(2271)) = (s(959)+s(2238)) = (s(1039)+s(2202)) = (s(1062)+s(2191)) = (s(1201)+s(2118)) = (s(1215)+s(2110)) = (s(1290)+s(2065)) = (s(1410)+s(1985)) = (s(1454)+s(1953)) = (s(1535)+s(1890)) = (s(1614)+s(1823)) = (s(1633)+s(1806)) = (s(1697)+s(1746))

***n* = 27**
35409725 = (s(85)+s(5950)) = (s(338)+s(5941)) = (s(650)+s(5915)) = (s(782)+s(5899)) = (s(826)+s(5893)) = (s(901)+s(5882)) = (s(994)+s(5867)) = (s(1339)+s(5798)) = (s(1547)+s(5746)) = (s(1675)+s(5710)) = (s(1790)+s(5675)) = (s(1973)+s(5614)) = (s(2086)+s(5573)) = (s(2210)+s(5525)) = (s(2443)+s(5426)) = (s(2597)+s(5354)) = (s(2725)+s(5290)) = (s(2875)+s(5210)) = (s(3029)+s(5122)) = (s(3094)+s(5083)) = (s(3466)+s(4837)) = (s(3502)+s(4811)) = (s(3563)+s(4766)) = (s(3638)+s(4709)) = (s(3835)+s(4550)) = (s(4069)+s(4342)) = (s(4165)+s(4250))

n = 32
29641625 = (s(67)+s(5444)) = (s(124)+s(5443)) = (s(284)+s(5437)) = (s(320)+s(5435)) = (s(515)+s(5420)) = (s(584)+s(5413)) = (s(835)+s(5380)) = (s(955)+s(5360)) = (s(1180)+s(5315)) = (s(1405)+s(5260)) = (s(1460)+s(5245)) = (s(1648)+s(5189)) = (s(1795)+s(5140)) = (s(1829)+s(5128)) = (s(1979)+s(5072)) = (s(2012)+s(5059)) = (s(2032)+s(5051)) = (s(2245)+s(4960)) = (s(2308)+s(4931)) = (s(2452)+s(4861)) = (s(2560)+s(4805)) = (s(2621)+s(4772)) = (s(2840)+s(4645)) = (s(3005)+s(4540)) = (s(3035)+s(4520)) = (s(3320)+s(4315)) = (s(3365)+s(4280)) = (s(3517)+s(4156)) = (s(3544)+s(4133)) = (s(3664)+s(4027)) = (s(3715)+s(3980)) = (s(3803)+s(3896))

MATH. Marvellous! And now it becomes interesting: can we discover a regularity? The first ten outcomes suggest that we can divide all numbers by 13, or, better, by 65. Moreover, do we have to distinguish between even and odd numbers? This becomes my task for the vacation period! Exciting!

COMP. Have a good time, Math! (*He leaves the room with a shrug of the shoulders*.)

LOG. (*Enters the room a few minutes later*.) Still working, Math?

MATH. Log gave me a tremendous amount of computations, and I am thinking them over.

LOG. Can I see them?

MATH. Here are the first ten items of Comp's list with the smallest numbers that are *n* times the sum of two squares. My outcome 325 is on this list:

***n* = 1**
5 = (s(1)+s(2))

***n* = 2**
65 = (s(1)+s(8)) = (s(4)+s(7))

***n* = 3**
325 = (s(1)+s(18)) = (s(6)+s(17)) = (s(10)+s(15))

***n* = 4**
1105 = (s(4)+s(33)) = (s(9)+s(32)) = (s(12)+s(31)) = (s(23)+s(24))

***n* = 5**
8125 = (s(5)+s(90)) = (s(27)+s(86)) = (s(30)+s(85)) = (s(50)+s(75)) = (s(58)+s(69))

***n* = 6**
5525 = (s(7)+s(74)) = (s(14)+s(73)) = (s(22)+s(71)) = (s(25)+s(70)) = (s(41)+s(62)) = (s(50)+s(55))

***n* = 7**
105625 = (s(36)+s(323)) = (s(80)+s(315)) = (s(91)+s(312)) = (s(125)+s(300)) = (s(165)+s(280)) = (s(195)+s(260)) = (s(204)+s(253))

***n* = 8**
27625 = (s(20)+s(165)) = (s(27)+s(164)) = (s(45)+s(160)) = (s(60)+s(155)) = (s(83)+s(144)) = (s(88)+s(141)) = (s(101)+s(132)) = (s(115)+s(120))

***n* = 9**
71825 = (s(1)+s(268)) = (s(40)+s(265)) = (s(65)+s(260)) = (s(76)+s(257)) = (s(104)+s(247)) = (s(127)+s(236)) = (s(160)+s(215)) = (s(169)+s(208)) = (s(188)+s(191))

***n* = 10**
138125 = (s(22)+s(371)) = (s(35)+s(370)) = (s(70)+s(365)) = (s(110)+s(355)) = (s(125)+s(350)) = (s(163)+s(334)) = (s(194)+s(317)) = (s(205)+s(310)) = (s(218)+s(301)) = (s(250)+s(275))

LOG. It's curious that there are no doubles. In fact, the smallest number that is the sum of two squares is not 5, but 2. For n = 2, I see immediately 50, because 50 = 49 + 1, *and* 50 = 25 + 25. Comp's solution gives the smallest numbers that are n times the sum of two *different* squares! I hope that it doesn't affect your solution for n = 3, Math!

MATH. (*After some scribbling on the blackboard.*) No, that's all right. Nevertheless, I will stay with Comp's outcomes, because they seem to have some nice properties. I factorized all reliable numbers and this resulted in the following table:

n = 1
$5 = 5$

n = 2
$65 = 5 \cdot 13$

n = 3
$325 = 5^2 \cdot 13$

n = 4
$1105 = 5 \cdot 13 \cdot 17$

n = 5
$8125 = 5^4 \cdot 13$

n = 6
$5525 = 5^2 \cdot 13 \cdot 17$

n = 7
$105625 = 5^4 \cdot 13^2$

n = 8
$27625 = 5^3 \cdot 13 \cdot 17$

n = 9
$71825 = 5^2 \cdot 13^2 \cdot 17$

n = 10
$138125 = 5^4 \cdot 13 \cdot 17$

n = 12
$160225 = 5^2 \cdot 13 \cdot 17 \cdot 29$

n = 13
$1221025 = 5^2 \cdot 13^2 \cdot 17^2$

n = 14

$3453125 = 5^6 \cdot 13 \cdot 17$

***n* = 15**
$1795625 = 5^4 \cdot 13^2 \cdot 17$
***n* = 16**
$801125 = 5^3.13.17.29$

***n* = 18**
$2082925 = 5^2 \cdot 13^2 \cdot 17 \cdot 29$

***n* = 20**
$4005625 = 5^4 \cdot 13 \cdot 17 \cdot 29$

***n* = 22**
$30525625 = 5^4 \cdot 13^2 \cdot 17^2$

***n* = 24**
$5928325 = 5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 37$

MATH. Wait, an email message from Comp. (*Math reads it.*)

LOG. What does Comp write?

MATH. He found more numbers, which he already factorized. I'll write them down, together with the value that we already have:

***n* = 27**
$5^2 \cdot 13^2 \cdot 17^2 \cdot 29$

***n* = 28**
$5^6 \cdot 13 \cdot 17 \cdot 29$

***n* = 30**
$5^4 \cdot 13^2 \cdot 17 \cdot 29$

***n* = 32**
$5^3 \cdot 13 \cdot 17 \cdot 29 \cdot 37$

***n* = 36**
$5^2 \cdot 13^2 \cdot 17 \cdot 29 \cdot 37$

***n* = 40**
$5^4 \cdot 13 \cdot 17 \cdot 29 \cdot 37$

***n* = 45**
$5^4 \cdot 13^2 \cdot 17^2 \cdot 29$

***n* = 48**
$5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41$

And, last but not least, surprisingly:

***n* = 11**
$5^{10} \cdot 13$

LOG. The powers of 5 are never less than those of the other prime numbers of the form $4k + 1$. Not all combinations occur. Do you have an explanation, Math?

MATH. I presume that the combinations which are skipped, belong to numbers with one of the required properties, without being minimal of course.

LOG. Let us try $5^2 \cdot 13^2$. How do we find the desired representations?

MATH. Believe it or not, the answer can be found on Internet, by what is called the computational knowledge machine Wolframalpha. (*Math goes to the site and he finds the answer*.) Look:

4225 has 4 representations as a sum of 2 squares:

$$4225 = 16^2 + 63^2 = 25^2 + 60^2 = 33^2 + 56^2 = 39^2 + 52^2$$

LOG. That's nice, and now $5^3 \cdot 13$.

1625 has 4 representations as a sum of 2 squares:

$$1625 = 5^2 + 40^2 = 16^2 + 37^2 = 20^2 + 35^2 = 28^2 + 29^2$$

MATH. Do you need more examples?

LOG. Perhaps $5^3 \cdot 13^2$ and $5^3 \cdot 13^3$?

MATH. (*He fills in 21125 and 274625*.)

21125 has 6 representations as a sum of 2 squares:

$$21\,125 = 10^2 + 145^2 = 26^2 + 143^2 = 31^2 + 142^2 = 65^2 + 130^2 = 79^2 + 122^2 = 95^2 + 110^2$$

274625 has 8 representations as a sum of 2 squares:

$$274\,625 = 7^2 + 524^2 = 65^2 + 520^2 = 140^2 + 505^2 = 191^2 + 488^2 = 208^2 + 481^2 = 260^2 + 455^2 = 320^2 + 415^2 = 364^2 + 377^2$$

LOG. Again a message from Comp. (*She opens it.*) Interesting! Comp had a brilliant insight! He found regularities by ordering the numbers in a special way, first 5, 5·17, 5·17·29, 5·17·29·37, 5·17·29·37·41, then 5^2, 5^2·17, 5^2·17·29, 5^2·17·29·37, and so on, each time taking the next power of 5 and omitting the factor 13. He encoded the prime factors by their exponents, with the numbers behind the minus signs giving the number of representations as sums of two different squares, and got the following results (*She prints them out.*)

5: 1 - 1
85: 101 - 2
2465: 1011 - 4
91205: 10111 - 8
3739405: 101111 - 16

25: 2 - 1
425: 201 - 3
12325: 2011 - 6
456025: 20111 - 12
18697025: 201111 - 24

125: 3 - 2
2125: 301 - 4
61625: 3011 - 8
2280125: 30111 - 16
93485125: 301111 - 32

625: 4 - 2
10625: 401 - 5
308125: 4011 - 10
11400625: 40111 - 20
467425625: 401111 - 40

3125: 5 - 3
53125: 501 - 6
1540625: 5011 - 12
57003125: 50111 - 24

15625: 6 - 3
265625: 601 - 7
7703125: 6011 - 14
285015625: 60111 - 28

MATH. It seems that the bare powers of 5 must be omitted in order to get complete regularities.

85: 101 - 2
2465: 1011 - 4

91205: 10111 - 8
3739405: 101111 - 16

425: 201 - 3
12325: 2011 - 6
456025: 20111 - 12
18697025: 201111 - 24

2125: 301 - 4
61625: 3011 - 8
2280125: 30111 - 16
93485125: 301111 - 32

10625: 401 - 5
308125: 4011 - 10
11400625: 40111 - 20
467425625: 401111 - 40

53125: 501 - 6
1540625: 5011 - 12
57003125: 50111 - 24

265625: 601 - 7
7703125: 6011 - 14
285015625: 60111 - 28

LOG. Comp missed two outcomes, namely 57003125·41 and 285015625·41. This seems to provide an opportunity for checking the rule that subjoining the next prime factor of the form $4k + 1$ leads to a doubling of the number of sums.

MATH. (*He does a little computing and consults Wolframalpha.*)

57003125·41 = 2337128125

2337128125 has 48 representations as a sum of 2 squares:

Promising!

285015625·41 = 11685640625

(*???*)

That's a pity, Wolfram gives no representations, although it mentions that 11685640625 is the hypotenuse of 16 primitive Pythagorean triples, but that's another topic. Perhaps the predicted number of 56 representations is too large for the computational knowledge machine? Still, the 48 representations of $5^5 \cdot 17 \cdot 29 \cdot 37 \cdot 41$ seem to give sufficient evidence for trusting the doubling rule. We must thank Comp![1] (*He writes a reply to Comp in which he congratulates him with what he achieved.*)

LOG. Nevertheless, I want a proof!

MATH. I will think it over ... and we have not even solved our original problem of finding regularities in the *minimal* numbers ... Perhaps a comparison between their table and Comp's last one will help.

LOG. It strikes me that the minimal numbers all have the factor 13, whereas this factor is completely absent in the regularities.

MATH. Comp arranged the numbers according to the powers of the factor 5 and the increasing series of factors of the form $4k + 1$. It seems that we must do the same with the minimals, to begin with. First the exponent 2, starting with the value for $n = 3$, my famous 325. I will write it as $m(3) = 325$. Then we get:

$m(3) = 5^2 \cdot 13$
$m(6) = 5^2 \cdot 13 \cdot 17$
$m(12) = 5^2 \cdot 13 \cdot 17 \cdot 29$
$m(24) = 5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 37$
$m(48) = 5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41$

LOG. I understand how it continues with 5^3:

$m(8) = 5^3 \cdot 13 \cdot 17$
$m(16) = 5^3 \cdot 13 \cdot 17 \cdot 29$
$m(32) = 5^3 \cdot 13 \cdot 17 \cdot 29 \cdot 37$

MATH. Isn't it simple? I continue:

$m(5) = 5^4 \cdot 13$
$m(10) = 5^4 \cdot 13 \cdot 17$
$m(20) = 5^4 \cdot 13 \cdot 17 \cdot 29$
$m(40) = 5^4 \cdot 13 \cdot 17 \cdot 29 \cdot 37$

[1] All credit to Dr. Jeroen Donkers, who not only wrote the computer programs, but also discovered the surprising theorem.

LOG. Yes, and finally:

$m(14) = 5^6 \cdot 13 \cdot 17$
$m(28) = 5^6 \cdot 13 \cdot 17 \cdot 29$

MATH. Let's now try the cases in which the factor 13 has the exponent 2, to begin with $n = 9$:

$m(9) = 5^2 \cdot 13^2 \cdot 17$
$m(18) = 5^2 \cdot 13^2 \cdot 17 \cdot 29$
$m(36) = 5^2 \cdot 13^2 \cdot 17 \cdot 29 \cdot 37$

LOG. Then we also have:

$m(15) = 5^4 \cdot 13^2 \cdot 17$
$m(30) = 5^4 \cdot 13^2 \cdot 17 \cdot 29$

MATH. So far, so good, but we skipped some values, not only 1, 2, and 4, but also 7, 13, and 22, not to mention 11. We can investigate what happens with them, consulting our friend Wolfram! The values of 1, 2, and 4 are 5, $5 \cdot 13$, and $5 \cdot 13 \cdot 17$. Let's see what happens with $5 \cdot 13 \cdot 17 \cdot 29$, that is, 32045.

32045 has 8 representations as a sum of 2 squares:

$$32\,045 = 2^2 + 179^2 = 19^2 + 178^2 = 46^2 + 173^2 = 67^2 + 166^2 = 74^2 + 163^2 = 86^2 + 157^2 = 109^2 + 142^2 = 122^2 + 131^2$$

LOG. That's OK, and wholly in the spirit of Comp! But 32045 isn't minimal, that was 27625. It's confirmed by Wolfram:

27625 is the smallest number with 8 representations as a sum of 2 squares:

$$27\,625 = 20^2 + 165^2 = 27^2 + 164^2 = 45^2 + 160^2 = 60^2 + 155^2 = 83^2 + 144^2 = 88^2 + 141^2 = 101^2 + 132^2 = 115^2 + 120^2$$

MATH. $m(7) = 5^4 \cdot 13^2$, but I don't know what to do with it. The next values are already given away, $5^4 \cdot 13^2 \cdot 17 = m(15)$, and $5^4 \cdot 13^2 \cdot 17^2 = m(22)$. It's true that we can go back from $m(14) = 5^6 \cdot 13 \cdot 17$ and $m(28) = 5^6 \cdot 13 \cdot 17 \cdot 29$, to get $5^6 \cdot 13$, and verify that it has 7 representations as a sum of 2 squares, but $5^6 \cdot 13$ is clearly larger than $5^4 \cdot 13^2$.

LOG. We know that $5^{10} \cdot 13$ has 11 representations, but is it minimal? Moreover, the relation with $m(22) = 5^4 \cdot 13^2 \cdot 17^2$ isn't clear. I'm afraid that we do not get any further. So far the results are impressive, but they leave much to be desired. And now I need a drink.

MATH. Wait a minute! The beautiful value for 11 gives me the idea that we should investigate all preceding numbers of the form $5^k{\cdot}13$ with the help of Wolfram, in so far as we don't already have their number of representations . I use the letter *n* in stead of *m*, because I expect that some values are not minimal.

(*By consulting Wolfram when necessary, Math comes to the following table.*)

$n(5^0{\cdot}13) = 1$
$n(5^1{\cdot}13) = 2$
$n(5^2{\cdot}13) = 3$
$n(5^3{\cdot}13) = 4$
$n(5^4{\cdot}13) = 5$
$n(5^5{\cdot}13) = 6$
$n(5^6{\cdot}13) = 7$
$n(5^7{\cdot}13) = 8$
$n(5^8{\cdot}13) = 9$
$n(5^9{\cdot}13) = 10$
$n(5^{10}{\cdot}13) = 11$

MATH. Now we see how Comp's solution for eleven representations is generated!

LOG. Amazing! It gives every number of representations! Maybe we can try to prove the general theorem, before going to the other regularities?

MATH. Seems plausible, but let's first take a drop.

(*He opens a cupboard and makes preparations for a drink.*)

LOG. If we stay in your room, we can investigate the $n(5^k{\cdot}13^2)$'s, the $n(5^k{\cdot}13^3)$'s, the $n(5^k{\cdot}13^4)$'s, and see how far we get on with them.

MATH. (*filling the glasses*) Go ahead!

LOG. (*a couple of minutes later, helped by Wolfram*) Here you are:

$n(5^0{\cdot}13^2) = 1$
$n(5^1{\cdot}13^2) = 3$

$n(5^2 \cdot 13^2) = 4$
$n(5^3 \cdot 13^2) = 6$

$n(5^4 \cdot 13^2) = 7$
$n(5^5 \cdot 13^2) = 9$

$n(5^6 \cdot 13^2) = 10$
$n(5^7 \cdot 13^2) = 12$

$n(5^8 \cdot 13^2) = 13$

$n(5^0 \cdot 13^3) = 2$
$n(5^1 \cdot 13^3) = 4$
$n(5^2 \cdot 13^3) = 6$
$n(5^3 \cdot 13^3) = 8$
$n(5^4 \cdot 13^3) = 10$
$n(5^5 \cdot 13^3) = 12$

$n(5^0 \cdot 13^4) = 2$

$n(5^1 \cdot 13^4) = 5$
$n(5^2 \cdot 13^4) = 7$

$n(5^3 \cdot 13^4) = 10$
$n(5^4 \cdot 13^4) = 12$

$n(5^5 \cdot 13^4) = 15$
$n(5^6 \cdot 13^4) = 17$

MATH. 17 representations in seven steps, incredible! This deserves a toast!

(*After quite a number of drinks, they end the session in high spirits, and go home.)*

(*When Log returns to the Institute after the holiday, the first thing she does is going to Math.)*

LOG. Happy new year, Math! Did you make any progress in the mean time?

MATH. I am still thinking about our extraordinary discovery of the generation of the natural numbers in the representations as a sum of two squares of the $n(5^k \cdot 13)$'s.

LOG. I see you didn't wipe it out. Did you find a proof? (*She points at the blackboard.)*

$$n(5^0 \cdot 13) = 1$$
$$n(5^1 \cdot 13) = 2$$
$$n(5^2 \cdot 13) = 3$$
$$n(5^3 \cdot 13) = 4$$
$$n(5^4 \cdot 13) = 5$$
$$n(5^5 \cdot 13) = 6$$
$$n(5^6 \cdot 13) = 7$$
$$n(5^7 \cdot 13) = 8$$
$$n(5^8 \cdot 13) = 9$$
$$n(5^9 \cdot 13) = 10$$
$$n(5^{10} \cdot 13) = 11$$

MATH. What bothers me is that I'm writing a book on productive problem solving in mathematics, and yet don't know how to handle the problem of proving the underlying theorem.

LOG. Did you consider writing 13, and powers of 5, as a sum of two irreducible squares?

MATH. Of course I did:

$13 = 2^2 + 3^2$

$5^1 = 1^2 + 2^2$
$5^2 = 3^2 + 4^2$
$5^3 = 2^2 + 11^2$
$5^4 = 7^2 + 24^2$
$5^5 = 38^2 + 41^2$
$5^6 = 44^2 + 117^2$
$5^7 = 29^2 + 278^2$
$5^8 = 336^2 + 527^2$

$5^9 = 718^2 + 1199^2$
$5^{10} = 237^2 + 3116^2$

Thereby a subproblem arose: finding a formula for 5^n as the sum of two irreducible squares. With the help of that formula we can redescribe $5^n \cdot 13$ in a possibly promising way.

LOG. What did you have in mind with these redescriptions?

MATH. That I will show you with 325, in other words, $5^2 \cdot 13$.

$$\begin{aligned} 5^2 \cdot 13 &= (3^2 + 4^2)(2^2 + 3^2) \\ &= 6^2 + 8^2 + 12^2 + 9^2 \\ &= 6^2 + 8^2 + 2 \cdot 8 \cdot 9 + 9^2 \\ &= 6^2 + (8 + 9)^2 \end{aligned}$$

Similarly,

$$\begin{aligned} &= 6^2 + 12^2 + 9^2 + 8^2 \\ &= 6^2 + 2 \cdot 6 \cdot 12 + 12^2 + 9^2 - 2 \cdot 9 \cdot 8 + 8^2 \\ &= (6 + 12)^2 + (9 - 8)^2 \end{aligned}$$

LOG. And how about the formula?

MATH. I couldn't find a formula for 5^n as the sum of two irreducible squares, but instead I found a connection between two successive sums. First I saw

$$5^7 = 29^2 + 278^2 \text{ and } 5^8 = 336^2 + 527^2$$

$$2 \cdot 278 - 29 = 527$$

Then I checked it with

$$5^8 = 336^2 + 527^2 \text{ and } 5^9 = 718^2 + 1199^2$$

Or perhaps can you do it?

LOG.

$$2 \cdot 527 = 1054 \text{ and } 1054 - 336 = 718$$

Remarkable!

MATH. But now look again at 5^7 and 5^8:

$$5^7 = 29^2 + 278^2 \text{ and } 5^8 = 336^2 + 527^2$$

$$2 \cdot 29 + 278 = 336$$
$$2 \cdot 278 - 29 = 527$$

LOG. I see, it's hardly necessary any longer to check it for 5^8 and 5^9:

$$5^8 = 336^2 + 527^2 \text{ and } 5^9 = 718^2 + 1199^2$$

$$2 \cdot 336 + 527 = 1199$$
$$2 \cdot 527 - 336 = 718$$

MATH. It's also possible to begin with the second part of 5^8:

$$2 \cdot 527 + 336 = 1390$$
$$2 \cdot 336 - 527 = 145$$

Only, this gives reducible squares. In general we have:

If

$$5^n = p^2 + q^2,$$

then

$$5^{n+1} = (2p + q)^2 + (2q - p)^2$$

and

$$5^{n+1} = (2q + p)^2 + (2p - q)^2$$

LOG. After all, this is almost trivial, but nevertheless it had still to be noticed. How now?

MATH. The trick with 325 didn't succeed for $n(5^k \cdot 13)$ with this result. Therefore I stopped.

LOG. Why not go straight to the sums of two squares of the $n(5^k \cdot 13)$'s themselves, starting with $5^1 \cdot 13$, in other words, 65:

$65 = 1^2 + 8^2$
$65 = 4^2 + 7^2$

I will apply the procedure that you derived for the powers of 5 on these outcomes, look:

$(2 \cdot 1 + 8)^2 + (2 \cdot 8 - 1)^2 = 10^2 + 15^2$
$(2 \cdot 1 - 8)^2 + (2 \cdot 8 + 1)^2 = 6^2 + 17^2$

$(2 \cdot 4 + 7)^2 + (2 \cdot 7 - 4)^2 = 15^2 + 10^2$
$(2 \cdot 4 - 7)^2 + (2 \cdot 7 + 4)^2 = 1^2 + 18^2$

MATH. It works! And we get one more representation for 325, or $5^2 \cdot 13$, than for $5^1 \cdot 13$, a reducible one, which can also be derived from $5^0 \cdot 13$. Let me see how it goes with the three representations of 325:

$(2 \cdot 10 + 15)^2 + (2 \cdot 15 - 10)^2 = 35^2 + 20^2$
$(2 \cdot 10 - 15)^2 + (2 \cdot 15 + 10)^2 = 5^2 + 40^2$

$(2 \cdot 6 + 17)^2 + (2 \cdot 17 - 6)^2 = 29^2 + 28^2$
$(2 \cdot 6 - 17)^2 + (2 \cdot 17 + 6)^2 = 5^2 + 40^2$

$(2 \cdot 1 + 18)^2 + (2 \cdot 18 - 1)^2 = 20^2 + 35^2$
$(2 \cdot 1 - 18)^2 + (2 \cdot 18 + 1)^2 = 16^2 + 37^2$

LOG. On balance, four representations of $5^3 \cdot 13$! Two irreducible, but also two reducible equations which we could have immediately derived from the representations of 65. They can be called 'reducible representations'.

MATH. OK. It's also clear that $10^2 + 15^2$, the reducible representation of $5^2 \cdot 13$, gives two reducible representations of $5^3 \cdot 13$. The irreducible representations give them too, but each of them gives an irreducible one, as well, or at least a representation that is not divisible by 5, and these two representations are different. It seems that the proof of the general theorem must follow on the same lines.

If

$$5^n \cdot 13 = p^2 + q^2,$$

then

$$5^{n+1} \cdot 13 = (2p + q)^2 + (2q - p)^2$$

and

$$5^{n+1} \cdot 13 = (2q + p)^2 + (2p - q)^2$$

LOG. Apparently, either $2p + q$ and $2q - p$ are both divisible by 5, or this holds for $2p - q$ and $2q + p$. How do you prove that?

MATH. Straightforward:

$p^2 + q^2 \equiv 0 \pmod 5$
$p^2 + 5pq + 6q^2 \equiv 0 \pmod 5$
$(p + 2q)(p + 3q) \equiv 0 \pmod 5$

Of course, either $p + 2q \equiv 0 \pmod 5$ or $p + 3q \equiv 0 \pmod 5$
The rest is child's play:

if $p + 2q \equiv 0 \pmod 5$, then also
$-4p + 2q \equiv 0 \pmod 5$ and $-2p + q \equiv 0 \pmod 5$;
then the second representation is reducible;

if $p + 3q \equiv 0 \pmod 5$, then also
$6p + 3q \equiv 0 \pmod 5$, and $2p + q \equiv 0 \pmod 5$

if $2p + q \equiv 0 \pmod 5$, then also
$2p - 4q \equiv 0 \pmod 5$, and $p - 2q \equiv 0 \pmod 5$;
then the first representation is reducible.

So it seems that the $n(5^k \cdot 13)$'s are no mystery any more.

LOG. A rigorous proof requires more.

MATH. I'm content with what we did. What I appreciate most, is that we found a method of construing more and more representations. It's a pity that Comp isn't with us, because he could easily write a program for it, whereas we must do it with paper and pencil, or chalk and blackboard!

LOG. I want to see just one more application. What do you think of $5^4 \cdot 13$?

MATH. We already have the four representations of $5^3 \cdot 13$:

$5^2 + 40^2$, $16^2 + 37^2$, $20^2 + 35^2$, $28^2 + 29^2$

$5^4 \cdot 13 = 50^2 + 75^2$ and $85^2 + 30^2$
$5^4 \cdot 13 = 69^2 + 58^2$ and $90^2 + 5^2$
$5^4 \cdot 13 = 75^2 + 50^2$ and $90^2 + 5^2$
$5^4 \cdot 13 = 85^2 + 30^2$ and $86^2 + 27^2$

Now we also have the five representations of $5^4 \cdot 13$:

$5^2 + 90^2$, $27^2 + 86^2$, $30^2 + 85^2$, $50^2 + 75^2$, $58^2 + 69^2$

LOG. And we know that $5^4 \cdot 13$ is the smallest number with exactly five representations as a sum of two squares. Well done! But now I'm going to work, see you Math.

MATH. Thanks a lot for your cooperation! Bye bye!

(*Math is still working on the subject when Comp enters the room.*)

COMP. Good morning, Math, and best wishes!

MATH. Thank you, the year began well, I found the following theorems:

If

$$5^n = p^2 + q^2,$$

then

$$5^{n+1} = (2p + q)^2 + (2q - p)^2$$

and

$$5^{n+1} = (2q + p)^2 + (2p - q)^2$$

and, with Log's help,

If

$$5^n \cdot 13 = p^2 + q^2,$$

then

$$5^{n+1} \cdot 13 = (2p + q)^2 + (2q - p)^2$$

and

$$5^{n+1} \cdot 13 = (2q + p)^2 + (2p - q)^2$$

COMP. I see, and this gave you the possibility to successively derive representations as a sum of two squares.

MATH. And now I try to generalize these results. I begin with bare powers:

If

$$c^n = p^2 + q^2 \text{ and } a^2 + b^2 = c,$$

then

$$c^{n+1} = (ap + bq)^2 + (aq - bp)^2$$

and

$$c^{n+1} = (aq + bp)^2 + (ap - bq)^2$$

COMP. Let me see how it works for 13.

$13 = 2^2 + 3^2$ and $2^2 + 3^2 = 13$
$13^2 = 13^2 + 0^2$
$13^2 = 12^2 + 5^2$

I continue with $13^2 = 12^2 + 5^2$
$13^3 = 39^2 + 26^2$
$13^3 = 46^2 + 9^2$, not bad!

$13^4 = 156^2 + 65^2$
$13^4 = 169^2 + 0^2$
$13^4 = 119^2 + 120^2$, nice!
$13^4 = 156^2 + 65^2$

It's better to work out the irreducible sums only:

$13^5 = (3\cdot119 + 2\cdot120)^2 + (3\cdot120 - 2\cdot119)^2$
$\quad = 597^2 + 122^2$ irreducible
$13^5 = (3\cdot120 + 2\cdot119)^2 + (3\cdot119 - 2\cdot120)^2$
$\quad = 598^2 + 117^2$, reducible

$13^6 = (3\cdot597 + 2\cdot122)^2 + (3\cdot122 - 2\cdot597)^2$
$\quad = 2035^2 + 828^2$

Interesting, but how do you want to proceed, when you successively hit upon 5, 13, 17, 29, 37, 41, that is, prime numbers of the form $4n + 1$?

MATH. Well, in that case the values of p and q change along with the next number of the row. For 5, they are 1 and 2, for 13, 2 and 3, for 17, 1 and 4, and so on.

COMP. Can you demonstrate it?

MATH. I take your list which begins with 325.

$m(3) = 5^2\cdot13$
$m(6) = 5^2\cdot13\cdot17$
$m(12) = 5^2\cdot13\cdot17\cdot29$
$m(24) = 5^2\cdot13\cdot17\cdot29\cdot37$
$m(48) = 5^2\cdot13\cdot17\cdot29\cdot37\cdot41$

We already know:

$5^2\cdot13 = 1^2 + 18^2$

$5^2 \cdot 13 = 6^2 + 17^2$
$5^2 \cdot 13 = 10^2 + 15^2$

Then we have the well-known decomposition of 17:

$17 = 4^2 + 1^2$

This determines the rule:

$5^2 \cdot 13 \cdot 17 = (4p + q)^2 + (4q - p)^2$

$5^2 \cdot 13 \cdot 17 = (4q + p)^2 + (4p - q)^2$

The application is very easy:

$5^2 \cdot 13 \cdot 17 = 22^2 + 71^2$
$5^2 \cdot 13 \cdot 17 = 73^2 + 14^2$
$5^2 \cdot 13 \cdot 17 = 22^2 + 71^2$
$5^2 \cdot 13 \cdot 17 = 41^2 + 62^2$
$5^2 \cdot 13 \cdot 17 = 64^2 + 7^2$
$5^2 \cdot 13 \cdot 17 = 55^2 + 50^2$
$5^2 \cdot 13 \cdot 17 = 70^2 + 25^2$

Do you want me to continue?

COMP. No, I'm convinced! The number of representations has been doubled, and I trust that the story is repeated when we compute the representations of $5^2 \cdot 13 \cdot 17 \cdot 29$. But I will show you that I understand your procedure, by calculating the first two sums, starting from $22^2 + 71^2$:

$(5 \cdot 22 + 2 \cdot 71)^2 + (5 \cdot 71 - 2 \cdot 22)^2 = 252^2 + 311^2$
$(5 \cdot 71 + 2 \cdot 22)^2 + (5 \cdot 22 - 2 \cdot 71)^2 = 399^2 + 32^2$

MATH. I think this will do for the moment.

COMP. I agree. Have a good day! *(He leaves the room, whereas Math is still thinking about the whole business.)*

MATH. (*talking to himself*) Why do we have to confine ourselves to different squares? Let me expand Log's example of 50. (*He goes to the blackboard.*):

$2 \cdot 5^2 = 5^2 + 5^2$
$50 \quad = 7^2 + 1^2$

$2 \cdot 13^2 = 13^2 + 13^2$
$338 \quad = 17^2 + 7^2$

$2 \cdot 17^2 = 17^2 + 17^2$
$578 \quad = 23^2 + 7^2$

$2 \cdot 29^2 = 29^2 + 29^2$
$1682 = 41^2 + 1^2$

We need a procedure for deriving representations of numbers with higher exponents, so it seems that we must modify my rule.

If

$$mc^n = p^2 + q^2 \text{ and } a^2 + b^2 = c,$$

then

$$mc^{n+1} = (ap + bq)^2 + (aq - bp)^2$$

and

$$mc^{n+1} = (aq + bp)^2 + (ap - bq)^2$$

Or else

If

$$mc^n = p_n^2 + q_n^2 \text{ and } a^2 + b^2 = c,$$

then

$$mc^{n+1} = (ap_n + bq_n)^2 + (aq_n - bp_n)^2$$

and

$$mc^{n+1} = (aq_n + bp_n)^2 + (ap_n - bq_n)^2$$

I start with the representation of $2 \cdot 5^0$.

$5 = 1^2 + 2^2$ — $a = 1$ and $b = 2$

$2 \cdot 5^0 = 1^2 + 1^2$ — $p_0 = 1$ and $q_0 = 1$

$2 \cdot 5^1 = 3^2 + 1^2$ — $p_1 = 3$ and $q_1 = 1$
$\quad = 3^2 + 1^2$

$2 \cdot 5^2 = 5^2 + 5^2$ — $p_2 = 5$ and $q_2 = 5$
$\quad = 7^2 + 1^2$ — $p_2 = 7$ and $q_2 = 1$

$2 \cdot 5^3 = 15^2 + 5^2$ — $p_3 = 15$ and $q_3 = 5$

$= 15^2 + 5^2$
$= 9^2 + 13^2$ $\qquad p_3 = 9$ and $q_3 = 13$
$= 15^2 + 5^2$

$2 \cdot 5^4 = 25^2 + 25^2$
$= 35^2 + 5^2$
$= 35^2 + 5^2$
$= 31^2 + 17^2$ $\qquad p_4 = 31$ and $q_4 = 17$

Humph, I am only interested in irreducible squares.

$2 \cdot 5^5 = 79^2 + 3^2$
$2 \cdot 5^6 = 161^2 + 73^2$
$2 \cdot 5^7 = 307^2 + 249^2$

The other way around?

$29^2 + 31^2 = 1802$

$$1802 = 2 \cdot 17 \cdot 53$$

$1802 = 34 \cdot 53$	$1802 = 106 \cdot 17$
$53 = 2^2 + 7^2$	$17 = 1^2 + 4^2$
$34 \cdot 53^0 = 3^2 + 5^2$	$106 \cdot 17^0 = 5^2 + 9^2$
$34 \cdot 53^1 = 41^2 + 11^2$ $= 31^2 + 29^2$	$106 \cdot 17^1 = 41^2 + 11^2$ $= 29^2 + 31^2$
$34 \cdot 53^2 = 159^2 + 265^2$ $= \mathbf{5^2 + 309^2}$ $= 265^2 + 159^2$ $= \mathbf{141^2 + 275^2}$	$106 \cdot 17^2 = 85^2 + 153^2$ $= \mathbf{175^2 + 3^2}$ $= 153^2 + 85^2$ $= \mathbf{147^2 + 95^2}$

Enough.

(*Math stops his investigations and turns to the preparation of a lecture. After a while Log enters the room.)*

LOG. I was thinking about 325, the smallest number that is three times the sum of two squares. You found it by trial and error, so to say. But why does it have that property? Without an answer to this question, we don't really understand what's going on!

MATH. I remember that Hans Hahn, the famous mathematician of the Vienna Circle, said something similar about mathematical proofs. We only understand a mathematical proof when we know why it goes as it goes and not otherwise. Well, I can reassure you. When we apply my theorem to the smallest incommensurable sums of two different squares, 5, as the sum of 1^2 and 2^2, and 13 as the sum of 2^2 and 3^2, then we get 5 as the smallest number that is the sum of two different squares, $5\cdot13$ as the smallest number that is twice the sum of two different squares, and one more with $5^2\cdot13$.

LOG. That is clear, but one is inclined to think that $5^3\cdot13$ is the smallest number that is the sum of four different squares. However, Wolfram gives $5\cdot13\cdot17$, as we know. I mention this example on purpose, because we missed $m(4)$ in our tables.

MATH. We can derive the sums from the representation of 17 and the two representations of 65 with my procedure:

$a = 1$ and $b = 4$

$p = 1$ and $q = 8$

$5\cdot13\cdot17 = 33^2 + 4^2$
$5\cdot13\cdot17 = 12^2 + 31^2$

$p = 4$ and $q = 7$

$5\cdot13\cdot17 = 32^2 + 9^2$
$5\cdot13\cdot17 = 23^2 + 24^2$

LOG. It follows that we can make the following mini-table:

$m(2) = 5\cdot13$
$m(4) = 5\cdot13\cdot17$

MATH. But $m(8) = 5^3\cdot13\cdot17$ according to Wolfram.

LOG. Look (*consulting Wolfram*):

27625 is the smallest number with 8 representations as a sum of 2 squares:

$$27625 = 20^2 + 165^2 = 27^2 + 164^2 = 45^2 + 160^2 = 60^2 + 155^2 = 83^2 + 144^2 = 88^2 + 141^2 = 101^2 + 132^2 = 115^2 + 120^2$$

There are only four irreducible representations, whereas all representations of $m(4)$ are irreducible. Let me see how it is with $5\cdot13\cdot17\cdot29$. (*She consults Wolfram again.*)

32045 has 8 representations as a sum of 2 squares:

$$32045 = 2^2 + 179^2 = 19^2 + 178^2 = 46^2 + 173^2 = 67^2 + 166^2 = 74^2 + 163^2 = 86^2 + 157^2 = 109^2 + 142^2 = 122^2 + 131^2$$

MATH. All are irreducible! That is the significance of my procedure! As a matter of fact, I am more interested in minimal irreducible representations than in minimal representations *tout court*.

LOG. Does this mean that we must start all over again?

MATH. Why not? We must make new tables:

$M(2) = 5 \cdot 13$
$M(4) = 5 \cdot 13 \cdot 17$
$M(8) = 5 \cdot 13 \cdot 17 \cdot 29$

Even my 325, $m(3)$ with small letter m, must be replaced.

LOG. This is too much for me! Who says that there is any solution at all for three irreducible representations, let alone for higher odd numbers? I quit. Have a good day! (*She leaves the room, whereas Math remains absorbed in thought.*)

(*However, after a while, Log returns.)*

MATH. You here again Log, what bothers you?

LOG. When I thought about your last problem, I tried some even numbers that are one larger than a square, because I considered your restriction to products of powers of prime numbers of the form $4n + 1$ too narrow. After all, my example of 50 was not that bad! See what I found:

$7^2 + 1^2 = 5^2 + 5^2$
$17^2 + 1^2 = 13^2 + 11^2$
$27^2 + 1^2 = 21^2 + 17^2$

$13^2 + 1^2 = 11^2 + 7^2$
$23^2 + 1^2 = 19^2 + 13^2$
$33^2 + 1^2 = 27^2 + 19^2$

Do you notice something?

MATH. Of course:

$(10n - 3)^2 + 1^2 = (8n - 3)^2 + (6n - 1)^2$

$(10n + 3)^2 + 1^2 = (8n + 3)^2 + (6n + 1)^2$

Very elementary equations, how could I have missed them! But I recognize good old Pythagoras in the coefficients! This makes a simplification possible:

$(5n - 3)^2 + 1^2 = (4n - 3)^2 + (3n - 1)^2$
$(5n + 3)^2 + 1^2 = (4n + 3)^2 + (3n + 1)^2$

LOG. Then we can also try other Pythagorean triples, for example (5, 12, 13). Let me try:

$(13n - x)^2 + 1^2 = (12n - y)^2 + (5n - z)^2$
$26x = 24y + 10z$

$13x = 12y + 5z$
$x^2 + 1 = y^2 + z^2$

$x^2 = y^2 + z^2 - 1$

$169x^2 = 169y^2 + 169z^2 - 169$
$144y^2 + 120yz + 25z^2 = 169y^2 + 169z^2 - 169$
$169 = 25y^2 - 120yz + 144z^2$
$(5y - 12z)^2 = 13^2$

$5y - 12z = 13$

$z = 1$
$y = 5$
$x = 5$

$(13n - 5)^2 + 1^2 = (12n - 5)^2 + (5n - 1)^2$

Another suitable sum-producing equation!

MATH. And also

$(13n + 5)^2 + 1^2 = (12n + 5)^2 + (5n + 1)^2$

It’s easy to generalize such equations for all Pythagorean triples.

Let $a^2 + b^2 = c^2$, then your formula $13x = 12y + 5z$ becomes $cx = by + az$ and $5y - 12z = 13$ becomes $ay - bz = c$. Take $z = 1$, then $y = (c + b)/a$ and $x = (c + b)/a$. Therefore the generalization is obvious:

If

$$a^2 + b^2 = c^2$$

then

$$(cn - (c + b)/a)^2 + 1^2 = (bn - (c + b)/a)^2 + (an - 1)^2$$
$$(cn + (c + b)/a)^2 + 1^2 = (bn + (c + b)/a)^2 + (an + 1)^2$$

LOG. It's a pity that the formulas still contain a constant. Can't we replace the 1's by expressions such as $(c + b)/a$?

MATH. I am afraid that then $(c + b)/a$ must also be replaced.

LOG. This means that we have to find new regularities, for example by looking at other numbers than my even multiples of 1. What do you think of 85?

MATH. It is twice the sum of two squares: $2^2 + 9^2 = 6^2 + 7^2$. But then it's interesting to consider more multiples of 5.

LOG. 125 for example? $2^2 + 11^2 = 5^2 + 10^2$.

MATH. Not very impressive, but now look at 130! $3^2 + 11^2 = 7^2 + 9^2$. Do you notice something?

LOG. Now we have

$2^2 + 9^2 = 6^2 + 7^2$
$3^2 + 11^2 = 7^2 + 9^2$

The first bases of the sums increase by 1, the second by 2. It would be nice if

$4^2 + 13^2 = 8^2 + 11^2$

MATH. It is: 185! And preceding $2^2 + 9^2 = 6^2 + 7^2$ we have your favorite 50:

$1^2 + 7^2 = 5^2 + 5^2$

LOG. The generalization is trivial:

$n^2 + (2n + 5)^2 = (n + 4)^2 + (2n + 3)^2$

MATH. Yes, but there is good old Pythagoras again!

LOG. This means that the formula for the next Pythagorean triple can be found by solving the following equation:

$n^2 + (xn + 13)^2 = (n + 5)^2 + (xn + 12)^2$

(*after some scribbling on the blackboard*) That's easy:

$n^2 + (5n + 13)^2 = (n + 5)^2 + (5n + 12)^2$

MATH. But what if we think of the general case, $a^2 + b^2 = c^2$?

LOG. Again solving an equation:

$n^2 + (xn + c)^2 = (n + a)^2 + (xn + b)^2$

(*Scribbling …*)

$n^2 + (an/(c - b) + c)^2 = (n + a)^2 + (an/(c - b) + b)^2$

Here you are! No constants any more! I am happy! (*She embraces Math.*)

MATH. This deserves a drink!

(*They leave the L. E. J. Brouwer Institute and go to an Amsterdam pub. Here the subject of their discussion is the problem of finding a general formula for numbers which are twice the sum of two cubic numbers, such as 4104 and Ramanujan's 1729. Blood is thicker than water!*)