POLYGONAL NUMBERS ADRIFT

LOG. (*standing in the doorway of Math's room*) Hallo Math, are you busy?

MATH. As always, but that does not mean that I have no time for you. Did you find something interesting?

LOG. (*entering the room*) Yes, I found a generalization of your favourite formula that says that the triangular number of a square is equal to the sum of the squares of two successive triangular numbers. Look (*she goes to the blackboard*):

$$t(s(n)) = s(t(n)) + s(t(n-1))$$

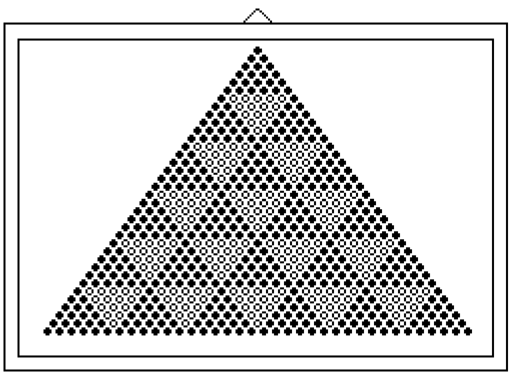
MATH. Do you mean that… (*he is interrupted by Log*)

LOG. Yes, a square is nothing but the product of two equal factors. The only thing we have to do is to take the product of two triangular numbers:

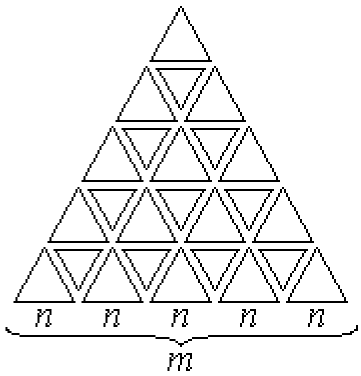
$$t(mn) = t(m)t(n) + t(m-1)t(n-1)$$

MATH. How did you find it?

LOG. Well, when I looked at your representation of the equation 666 = 441 + 225 (*she points at the picture on the wall*),



then I realized that it is not necessary to take six triangles on its base, we can take any number of triangles we want (*she makes the sketch shown below)*:



MATH. I see. There is no need to explain it to me any further. But how is it possible that I did not see it when I found my formula? It is amazing! I am much obliged to you. It also seems that you just demonstrated that perspicuous representations may have a heuristic use. Moreover, your new figure is also convincing in the sense that it makes a formal algebraical proof superfluous. Or do you still want to show that

$$\tfrac{1}{2}mn(mn + 1) = \tfrac{1}{2}m(m + 1)\cdot\tfrac{1}{2}n(n + 1) + \tfrac{1}{2}(m - 1)m\cdot\tfrac{1}{2}(n - 1)n$$

LOG. Not at all, although I think that this would be a nice exercise for high school students. By the way, do you know the formula for the triangular number of a sum?

MATH. Do you mean

$$\mathrm{t}(m + n) = \mathrm{t}(m) + mn + \mathrm{t}(n)$$

LOG. Yes, and I found also a generalization of this formula. My idea was simple; I first considered the formula for squares, in your notation:

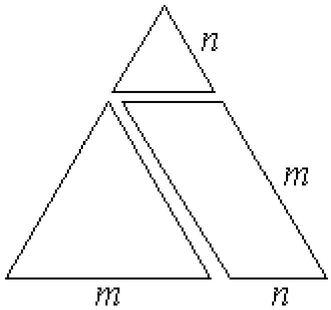
$$\mathrm{s}(m + n) = \mathrm{s}(m) + 2mn + \mathrm{s}(n)$$

MATH. Aha, then the general formula for a polygonal number of a sum becomes:

$$\mathrm{p}_k(m + n) = \mathrm{p}_k(m) + (\mathrm{k} - 2)mn + \mathrm{p}_k(n)$$

I do not need to see the algebraical proof. I will leave that to my nephew Arit, whom I am helping with his algebra. But do you perhaps have a perspicuous representation of this formula?

LOG. Yes, I do. I started with the formula for t($m + n$), or, as you now call it, $\mathrm{p}_3(m + n)$, and drew the following picture:



MATH. That is easy, but do the standard representations of higher polygonal numbers also allow such simple configurations?

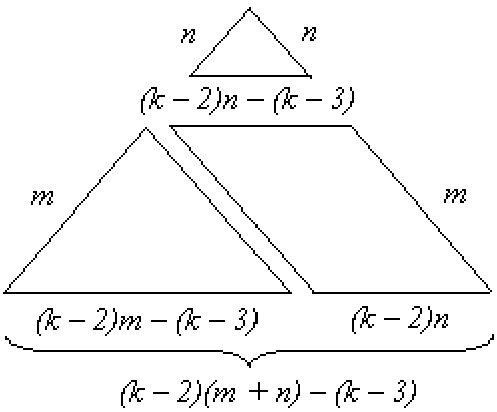
LOG. No, but I discovered that they can get triangular representations as well. This goes as follows. Consider the formula for a polygonal number in general:

$$p_k(n) = \tfrac{1}{2}n((k-2)n-(k-4))$$

This can be seen as the formula for the sum of an arithmetical series of n terms, with first term 1 and last term

$$(k-2)n-(k-3)$$

But now that we know the last term we can represent $p_k(m+n)$ as a figure consisting of two triangles and a parallelogram too:



Look, $(k-2)n$ is the difference of $(k-2)(m+n)-(k-3)$ and $(k-2)m-(k-3)$ and this explains, in a sense, that $(k-2)mn$ occurs in the formula for $p_k(m+n)$.

MATH. Very nice! I am impressed! Especially your redescription of the formula for $p_k(n)$ is important as an example of a productive problem solving procedure, if I may say so. I do not believe that a computer could do this.

COMP. (*appearing in the door opening*) Speak of the devil and his imp appears. Were you speaking ill of me?

MATH. (*laughing*) I am surprised that you used a quotation from *Intuitionism*. Are you reading it?

COMP: Yes, I have been told that some computer scientists pretend to be intuitionists in matters of philosophy of mathematics, so I decided to study Heyting's book and I just arrived at page 5, you see. But what does that figure on the blackboard mean? Can you explain it to me? A computer programmer should at least know what the problem solution is, before he starts thinking about how a computer could find it.

LOG. Go ahead, Math. After all you are a trained mathematics teacher!

MATH. Oh please! Comp, this time we were discussing polygonal numbers in general, and that is why there are formulas in the picture. But I can give you a concrete example, to begin with. You know how triangular numbers appear as sums, for example

$$t(5) = 1 + 2 + 3 + 4 + 5$$

Now we do not use the letter 't' anymore, but replace it by 'p_3', for obvious reasons. I trust that you now understand the following formula:

$$p_3(n) = 1 + 2 + 3 + 4 + \ldots + n$$

COMP. No problem.

MATH. The next formula concerns squares. Look:

$$p_4(n) = 1 + 3 + 5 + 7 + \ldots + n$$

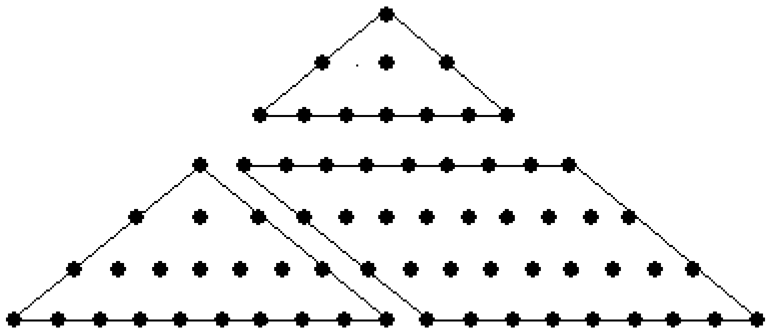
COMP. I see. May I write down the formula for pentagonal numbers?

MATH. Go ahead!

COMP.

$$p_5(n) = 1 + 4 + 7 + 10 + \ldots + n$$

MATH Let me now draw a picture of $p_5(7)$ and locate three groups in it:



What do you notice?

COMP. $p_5(7)$ consists of $p_5(3)$ and $p_5(4)$ and, eh ... 36 other points arranged in a parallelogram, just as in the picture above.

MATH. The problem was to find a formula for the other points. What do you think?

COMP. I see. Every pentagonal number is the sum of an arithmetical progression. We need the formula for the last term if we want to compute the length of the parallelogram.

MATH. Just do it!

COMP.

$$p_5(n) = 1 + (1 + 3) + (1 + 3 + 3) + (1 + 3 + 3 + 3) + \ldots + (! + \underbrace{3 + 3 + 3 + \ldots + 3)}_{n-1}$$

The formula for the last term is $1 + 3(n - 1)$. The length of the parallelogram is in general

$$1 + 3(m + n - 1) - (1 + 3(m - 1)) = 3n$$

Now I know at least that (*looking at the figure that Math drew*):

$$p_5(m + n) = p_5(m) + p_5(n) + 3mn$$

This fits in with your figure; 36 is equal to 3 times 3 times 4. Oh, for $p_k(m + n)$ the coefficient of mn must be $k - 2$., so (*with a certain triumph*):

$$p_k(m + n) = p_k(m) + p_k(n) + (k - 2)mn$$

MATH. Excellent!

LOG. I agree, but in fact this is only *Spielerei* without scientific significance; or did you have some interesting novelty, Math?

MATH. Only something that supports my ideas about intuitive insights. But maybe it is interesting for Comp too, for I wonder whether a computer program could find a certain formula.

COMP. That was just the reason that I came to see you. I was anxious to hear if you had some work for me. You know; I still believe that computers can do more than you are inclined to think.

LOG. Well, Math, that is a challenge!

MATH. All right. I was also interested in generalizations of earlier results and I found a formula for triangular numbers of polygonal numbers in general. However, later on I gave up my fixation on triangular numbers and then I found a simple formula for

$$p_\lambda(p_\mu(n)) - p_\mu(p_\lambda(n))$$

It is on this point that I thought of Comp. Would his computer find my formula? But I will first sketch how I found a special formula for

$$p_3(p_k(n))$$

It is again a question of intuitive insights which mathematicians are good at, although it is still an unsettled problem why our spontaneous solutions are oftener right than wrong as Russell said. Yet I will give a suggestion in which direction this problem might be solved.

LOG. I assume that you did not work out your last expression with the help of the general formula for a polygonal number?

MATH. No, I started with a table. Here it is:

n	$p_3(n)$	$p_4(n)$	$p_5(n)$	$p_4(p_3(n))$	$p_3(p_4(n))$	$p_5(p_3(n))$	$p_3(p_5(n))$
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	3	4	5	9	10	12	15
3	6	9	12	36	45	51	78
4	10	16	22	100	136	145	253
5	15	25	35	225	325	330	630

I already knew that each number of the sixth column is equal to the sum of the last two numbers of the preceding column, but I saw that this rule does not hold for 630, the last number of the last column. So I tried a new rule and took 330 and two times 145. That gave 620 and that is almost 630. It is only 10 less. Then I applied the new rule to 253 and took 145 and two times 51. That gave 247 and that is only 6 less than 253. Well, 10 and 6 are very familiar to us, so I did not hesitate to write down the following formula:

$$p_3(p_5(n)) = p_5(p_3(n)) + 2p_5(p_3(n-1)) + p_3(n-1)$$

In order to generalize this formula I enlarged the table with columns for $p_6(p_3(n))$ and $p_3(p_6(n))$ as follows:

$p_6(p_3(n))$	$p_3(p_6(n))$
0	0
1	1
15	21
66	120
190	4 06
435	1035

Well, to make a long story short, I found

$$p_3(p_6(n)) = p_6(p_3(n)) + 3p_6(p_3(n-1)) + 3p_3(n-1)$$

and then I listed the four formulas that I now had:

$$p_3(p_3(n)) = p_3(p_3(n))$$
$$p_3(p_4(n)) = p_4(p_3(n)) + 1p_4(p_3(n-1))$$
$$p_3(p_5(n)) = p_5(p_3(n)) + 2p_5(p_3(n-1)) + 1p_3(n-1)$$
$$p_3(p_6(n)) = p_6(p_3(n)) + 3p_6(p_3(n-1)) + 3p_3(n-1)$$

What do you think?

LOG. That the coefficients of $p_3(n-1)$ are again triangular numbers?

MATH. Indeed. The result is clear:

$$p_3(p_k(n)) = p_k(p_3(n)) + (k-3)p_k(p_3(n-1)) + p_3(k-4)p_3(n-1)$$

COMP. Did you check this formula?

MATH. As a matter of fact, yes. I made two more columns:

$p_7(p_3(n))$	$p_3(p_7(n))$
0	0
1	1
18	28
81	171
235	595
540	1540

COMP. Let me see, 540 plus four times 235 makes 1480. $p_3(3) = 6$ and $p_3(4) = 10$, that gives 60, 1480 plus 60 gives 1540. It seems that you are right!

LOG. Did you prove your formula?

MATH. Yes, I did, but the strange thing was that at first it did not come out. However, I never doubted that my formula would be correct, and I sought the mistake in my algebraical derivation. And I was right.

LOG. Last week you quoted Russell, who wrote in his book *Human Knowledge* that he did not know how to make explicit what guides mathematical intuition in such cases as you dealt with. A few moments ago you promised to give a hint for an answer to Russell's question.

MATH. All my examples made use of polygonal numbers. They gave rise to simple functions. When I solved my problems about these functions, I found equally simple relationships; more precisely, I only made use of polygonal functions themselves. Look at the coefficients of $p_3(n-1)$ in the list above. They were 1 and 3. Now someone might think that they should be seen as the first two odd numbers instead of the first two triangular numbers. But as soon as it was found that the third coefficient was 6, this possibility was ruled out. Yet it remains the case that

mathematical intuition is also guided by the discovery of conspicuous connections.
LOG. It is only a pity that your last example is so unimportant. Moreover, the resulting formula is not interesting.

MATH. I agree. However, I would not have found my next attractive formula without it! That is to say, we will soon be able to overturn the ladder, to quote Sextus Empiricus. But before doing that, I will derive a new table from the tables that we already have. The reason for it will become clear afterwards. What I would like you to do for me Comp, is to let your computer find a formula, without further information. In the mean time, I will tell Log my solution.

COMP. That's agreed!

MATH: Here is the derived table:

$f(3,3) = 0$	$f(4,4) = 0$	$f(5,5) = 0$	$f(6,6) = 0$
$f(3,4) = 1$	$f(4,5) = 3$	$f(5,6) = 6$	$f(6,7) = 10$
f 3,5) = 3	$f(4,6) = 8$	$f(5,7) = 15$	$f(6,8) = 24$
$f(3,6) = 6$	$f(4,7) = 15$	$f(5,8) = 27$	$f(6,9) = 42$

COMP. I understand that you want to know whether a computer can find an algebraical expression for this function. I will do my best. But please give me also a sheet of paper with the original table. I want to see whether my computer can find a relationship between the values in the last column and preceding ones.

(*Math looks in his papers and takes out two sheets of paper and gives one to Comp. Comp leaves the room.*)

....

MATH. (*shows the other piece of paper to Log*) Look, Log:

n	$p_4(p_3(n))$	$p_3(p_4(n))$	$p_5(p_3(n))$	$p_3(p_5(n))$	$p_6(p_3(n))$	$p_3(p_6(n))$	$p_7(p_3(n))$	$p_3(p_7(n))$
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	9	10	12	15	15	21	18	28
3	36	45	51	78	66	120	81	171
4	100	136	145	253	190	406	235	595
5	225	325	330	630	435	1035	540	1540

When I arrived at this table, I noticed that the differences between the pairs in the last line are always a multitude of 100. I also saw that the differences in the last line but one are a multitude of 36. This suggested a formula of the following form:

$$p_\lambda(p_\mu(n)) - p_\mu(p_\lambda(n)) = f(\lambda,\mu)p_4(p_3(n-1))$$

The problem was now, to determine the function f. That is why I asked Comp to find this function too, although I drew up the derived table only after I had found it myself. I did this as follows. I noticed that

$$f(\lambda,\mu) = p_\lambda(p_\mu(2)) - p_\mu(p_\lambda(2))$$

The values of $p_\mu(2)$ and $p_\lambda(2)$ are, of course, easy to compute with the formula for $p_k(n)$:

$$p_k(n) = \tfrac{1}{2}n((k-2)n - (k-4))$$

LOG. Let me see:

$$p_\mu(2) = \tfrac{1}{2}\cdot 2((\mu-2)2 - (\mu-4)) = \mu$$

That is nice! Then $p_\lambda(2)$ must be λ. We are almost ready; we have only to apply the formula for $p_k(n)$:

$$\begin{aligned}
f(\lambda,\mu) &= p_\lambda(p_\mu(2)) - p_\mu(p_\lambda(2)) \\
&= p_\lambda(\mu) - p_\mu(\lambda) \\
&= \tfrac{1}{2}\,\mu((\lambda-2)\mu - (\lambda-4)) - \tfrac{1}{2}\,\lambda((\mu-2)\,\lambda - (\mu-4)) \\
&= \tfrac{1}{2}(\mu^2\lambda - 2\mu^2 - \mu\lambda + 4\mu - \lambda^2\mu + 2\lambda^2 + \lambda\mu - 4\lambda) \\
&= \tfrac{1}{2}(\mu^2\lambda - 2\mu^2 + 4\mu - \lambda^2\mu + 2\lambda^2 - 4\lambda) \\
&= \tfrac{1}{2}(\mu\lambda(\mu-\lambda) - 2(\mu^2 - \lambda^2) + 4(\mu-\lambda)) \\
&= \tfrac{1}{2}(\mu-\lambda)(\mu\lambda - (\mu+\lambda) + 4) \\
&= \tfrac{1}{2}(\mu-\lambda)(\mu-2)(\lambda-2)
\end{aligned}$$

That is an amazingly simple formula!

Now we can also rewrite $p_4(p_3(n-1))$ as $(\tfrac{1}{2}n(n-1))^2$, and your general formula becomes:

$$p_\lambda(p_\mu(n)) - p_\mu(p_\lambda(n)) = \tfrac{1}{2}(\mu-\lambda)(\mu-2)(\lambda-2)(\tfrac{1}{2}n(n-1))^2$$

or something like that.

MATH. Excellent! But I wonder whether the algebraical derivation of the formula for $f(\lambda,\mu)$ cannot also be found intuitively from the derived table:

$f(3,3) = 0$	$f(4,4) = 0$	$f(5,5) = 0$	$f(6,6) = 0$
$f(3,4) = 1$	$f(4,5) = 3$	$f(5,6) = 6$	$f(6,7) = 10$
$f(3,5) = 3$	$f(4,6) = 8$	$f(5,7) = 15$	$f(6,8) = 24$
$f(3,6) = 6$	$f(4,7) = 15$	$f(5,8) = 27$	$f(6,9) = 42$

Let me show you what such an intuitive derivation could look like!

LOG. Do you mean that you yourself did not find the formula in that way?

MATH. You are right, and it is easy to present such a derivation with hindsight. This does not imply that it is far-fetched. Look, because the value of $f(\lambda,\mu)$ is 0 if λ and μ are equal, it is divisible by $\mu - \lambda$. Notice that I assume that μ is greater than λ. It follows that $f(\lambda,\mu)$ has the form

$$(\mu - \lambda)g(\lambda,\mu)$$

Now we can make the corresponding table for $g(\lambda,\mu)$ as follows:

$g(3,3) = 0$	$g(4,4) = 0$	$g(5,5) = 0$	$g(6,6) = 0$
$g(3,4) = 1$	$g(4,5) = 3$	$g(5,6) = 6$	$g(6,7) = 10$
$g(3,5) = \frac{1}{2}\cdot 3$	$g(4,6) = 4$	$g(5,7) = \frac{1}{2}\cdot 15$	$g(6,8) = 12$
$g(3,6) = 2$	$g(4,7) = 5$	$g(5,8) = 9$	$g(6,9) = 14$

Or, if we make it more uniform,

$g(3,3) = 0$	$g(4,4) = 0$	$g(5,5) = 0$	$g(6,6) = 0$
$g(3,4) = \frac{1}{2}\cdot 2$	$g(4,5) = \frac{1}{2}\cdot 6$	$g(5,6) = \frac{1}{2}\cdot 12$	$g(6,7) = \frac{1}{2}\cdot 20$
$g(3,5) = \frac{1}{2}\cdot 3$	$g(4,6) = \frac{1}{2}\cdot 8$	$g(5,7) = \frac{1}{2}\cdot 15$	$g(6,8) = \frac{1}{2}\cdot 24$
$g(3,6) = \frac{1}{2}\cdot 4$	$g(4,7) = \frac{1}{2}\cdot 10$	$g(5,8) = \frac{1}{2}\cdot 18$	$g(6,9) = \frac{1}{2}\cdot 28$

As soon as someone discovers that the numbers in the second column are all divisible by 2, and those in the third column by 3, and the numbers in the fourth column by 4, he can rewrite this table with the following result:

$g(3,3) = 0$	$g(4,4) = 0$	$g(5,5) = 0$	$g(6,6) = 0$
$g(3,4) = \frac{1}{2}\cdot 2$	$g(4,5) = \frac{1}{2}\cdot 2\cdot 3$	$g(5,6) = \frac{1}{2}\cdot 3\cdot 4$	$g(6,7) = \frac{1}{2}\cdot 4\cdot 5$
$g(3,5) = \frac{1}{2}\cdot 3$	$g(4,6) = \frac{1}{2}\cdot 2\cdot 4$	$g(5,7) = \frac{1}{2}\cdot 3\cdot 5$	$g(6,8) = \frac{1}{2}\cdot 4\cdot 6$
$g(3,6) = \frac{1}{2}\cdot 4$	$g(4,7) = \frac{1}{2}\cdot 2\cdot 5$	$g(5,8) = \frac{1}{2}\cdot 3\cdot 6$	$g(6,9) = \frac{1}{2}\cdot 4\cdot 7$

Then the rest is child's play. Yet I must confess that I gave the derived table as an exercise to several mathematicians. All of them succeeded in finding the solution, but none of them did it in the intuitive way. I am anxious to hear if Comp, or rather his computer found it.

COMP. (*entering the room*) Here is the imp again! His devilish powers have overcome!

MATH. Welcome Comp. What did you find?

COMP. First of all, I wrote a program for your original table. In order to find a connection between the numbers in the last column and preceding values, I considered only numbers in the same row. Moreover, I restricted the absolute value of the coefficients to a maximum of 3. Nevertheless the computer found 48 solutions within the total number of combinations that was eh… (*Comp shows the list with the solutions to Math.*)

MATH. Ah, there are two solutions with only three values, in your notation:

$$[0, -1, 0, 0, 0, 3, -1]$$

and

$$[0, 0, 0, 0, -3, 3, 1]$$

This means that

$$p_3(p_5(n)) = 3p_3(p_4(n)) - p_5(p_3(n)) - p_3(n)$$

and

$$p_3(p_5(n)) = p_5(p_3(n)) + 3p_3(p_4(n)) - 3p_4(p_3(n))$$

The second formula can be rewritten as

$$p_3(p_5(n)) - p_5(p_3(n)) = 3(p_3(p_4(n)) - p_4(p_3(n)))$$

and that is interesting, because it is a consequence of our general formula.

COMP. I understand that your general formula has something to do with the function of two variables – in my notation X and Y – that you asked me to find. Well, I also wrote a program for that problem. I tried evolutionary algorithms on expression trees, with operations +, –, and *, plus constants from -5 to 5. And indeed my computer came up with a solution, after hours of searching. Here it is (*Comp shows Math a piece of paper with the following formula*):

(((5 * (((X * ((((0 * Y) * (Y + Y)) + (Y * (X – -4))) + X)) * X) + (Y – (((Y + X) – ((X * Y) – ((5 * Y) + 4))) + (((3 + (Y + X)) – ((-3 – -2) * (-2 + X))) + (((X – Y) * X)))))) – (1 * (Y – Y))) + (Y – ((((4 * 5) * Y) – Y) * (X * (Y + ((Y + X) * ((3 * (-4 – 2)) + (((X – X) + (Y + X)) * Y))))))))

This did not look not very elegant to me, so I simplified it to (*Comp writes on the blackboard*):

(19Y^2 + 5Y + 5)X^3 + (38Y^3 – 322Y – 5)X^2 + (19Y^4 – 361Y^2 + 5Y – 15)X –29Y – 25

Is this what you mean?

LOG. (*laughing*) Do say yes Math!

MATH. With pleasure, but I cannot believe that the last formula is correct. Did you check your solution, Comp?

COMP. No, I did not, but I am not content with it anyway. The formula is too complicated. But what else can I do?
MATH. Maybe you can add the division by 2, and also restrict the other coefficients and constants to -2, -1, 1 and 2. I am curious…

COMP. OK. (*leaves the room*)

(*While Comp is working on his task, Log and Math begin to check Comp's first formula. First of all they give X and Y the values 4 and 5, but they do not get the corresponding function value 3. Then they try again with the arguments 3 and 4, with the same negative result. They conclude that the formula must be wrong. Comp returns almost an hour later.*)

COMP. My computer triumphed! After adding your restrictions, some debugging and some tuning, my computer found the following formula:

(((Y – 2)*((X – Y)*(Y – (X + (Y – 2)))))/2)

I simplified it to:

(Y – X)*(X – 2)*(Y – 2)/2

Seems correct to me.

LOG. Did you expect that, Math?

MATH. Yes, in a way. Nevertheless, I admire your achievement, Comp. With hindsight I think that computers can be of great heuristic help, provided that they are instructed in advance with constraints such as a limitation of the coefficients to small numbers. Take your second formula after I had rewritten it:

$$p_3(p_5(n)) - p_5(p_3(n)) = 3(p_3(p_4(n)) - p_4(p_3(n)))$$

It is possible that my investigation had taken a different course if I had known this formula. On the other hand, your first formula appeared to be incorrect. It seems that you made a mistake when you copied it.

COMP. That is possible. Errare humanum est.

LOG. Quite right. How often do we not make mistakes, not only when we are guessing, but even when we are applying formulas in a mechanical way.

MATH. I agree. The question is: which do you trust more, your formal skills or my intuitive abilities?

COMP. Or my computer power?

(*The discussion ends in laughter*.)