PERSPICUOUS REPRESENTATIONS IN ALGEBRA

LOG. More than once you draw attention to perspicuous representations. Do you also count as such the pictures which sometimes accompany algebraic formulas?

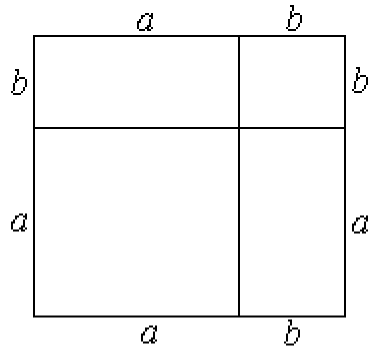
MATH. Perhaps we must go back to the time before the invention of algebra. That is to say, the ancient Greeks did ‘know’ several algebraic theorems, but they formulated them geometrically. For them, our formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

was worded as – I will give a quotation from a book on ‘the life of boys’ in ancient Athens:

> When a straight line segment is divided at any point in two parts, then the square on the whole segment will be equal in area as the sum of the squares on its two parts, augmented by the double of the rectangle which has the two parts as its sides.[1]

This means that the ancient mathematicians had in mind a picture like the following:



I think that you will agree this picture is more ‘perspicuous’ than the above-given verbal description, under the assumption that one knows how to compute the area of a rectangle given the lengths of its sides. It is even possible that it once served as a proof of a rule for the computation of the square of the sum of two arbitrary numbers.

[1] After K. Kuiper, *Atheens jongensleven*. Haarlem: Tjeenk Willink & Zoon, 1911, p. 104

However, the Babylonians knew already general relationships between numbers without the need of a geometrical representations. For example, they could find two numbers with a given sum, say 20, and product, say 91, by first computing their average , that is half of the sum, here 10. Then they computed the square of this average, 100, and next they computed its ‘deviation’ from the product by taking the difference, in this case 9. Then they computed the square root of this difference – ‘what times what makes 9?’ – and thereby they knew the deviation, obviously 3, of the numbers from their average. They are 10 – 3 and 10 + 3, in other words, 7 and 13. They could also do this with arbitrarily chosen sums and products which do not lead to whole numbers. Hoewever, they could not algebraically prove the correctness of this procedure, and I have not seen any geometrical demonstration.

COMP. To us, the algebraic formulas are simple:

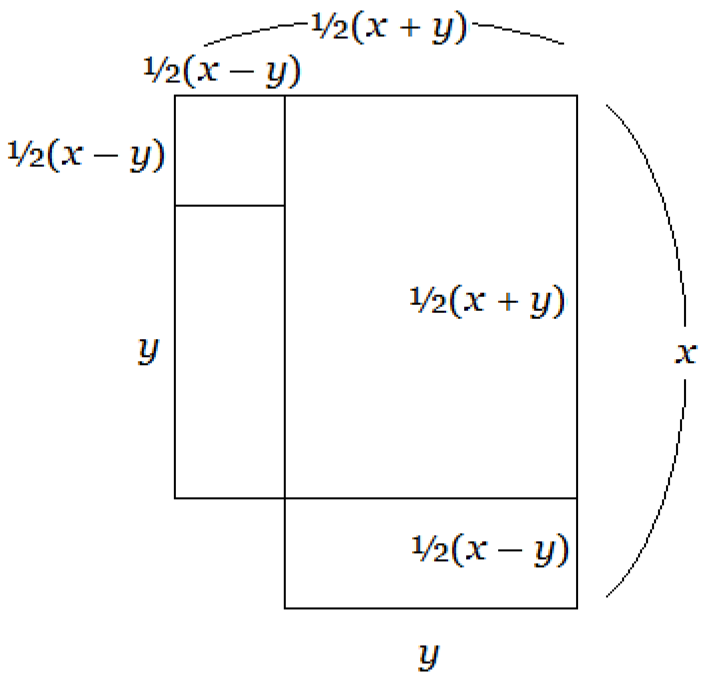
$$(½(x + y))^2 - xy = (½(x - y))^2$$

$$½(x + y) + ½(x - y) = x$$

$$½(x + y) - ½(x - y) = y$$

But how would a picture of the first formula look like?

MATH.



COMP. I see. It is a special case of a picture for $(a + b)(a - b) = a^2 - b^2$.

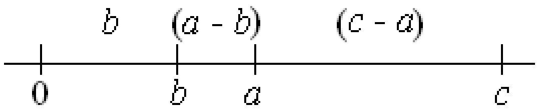
In order to get this back, we need only the following substitutions:

$$\tfrac{1}{2}(x + y) := a$$
$$\tfrac{1}{2}(x - y) := b$$
$$x := (a + b)$$
$$y := (a - b)$$

LOG. I find the following linguistic formulation of the corresponding 'theorem' already sufficiently perspicuous:

> The deviation of a product of two numbers from the square of their average is equal to the square of their deviations from that average.

MATH. Things are different with more elementary algebraic formulas, as we know from Felix Klein's illustration of the algebraic theorem $c - (a - b) = (c - a) + b$ by means of points on the number line:[2]



LOG. It seems useful that pupils themselves first exercise with simple examples such as Jordanus' formula $c - b = (c - a) + (a - b)$ for $c > a > b$, and then determine the segments that form $c - (a - b)$ and $(c - a) + b$. Then they 'see' that the formula is correct.

MATH. Yes. Klein went even so far that he wrote:

> From the present point of view, we have the so-called parenthesis rules for operations with positive numbers, which are, of course, continued in our fundamental formulas, provided one includes the corresponding laws for subtraction. But I should take them up somewhat in detail, by means of two examples, in order, above all, to show the possibility of extremely simple intuitive proofs for them, proofs which consist only of the representation and the word 'Look'!, as was the custom with the ancient Hindus.[3]

However, his second example was the formula

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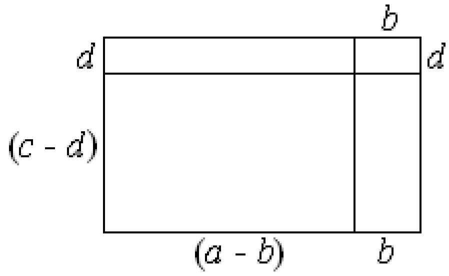
[2] Felix Klein, *Elementarmathematik vom höheren Standpunkte aus*. Vierte Auflage. Erster band. *Arithmetik. Algebra. Analysis*. Ausgearbeitet von E. Hellinger. Für den Druck fertig gemacht und mit Zusätzen versehen von Fr. Seyfarth. Berlin: Julius Springer, 1933, p. 28.

[3] Felix Klein, *Elementary mathematics from an advanced standpoint. Arithmetic. Algebra. Analysis.* Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York: Dover Publications, 1945, p. 26.

$$(a - b)(c - d) = ac - ad - bc + bd$$

together with a picture in the style of the ancient Greeks:



LOG. My objection to the idea that such pictures may serve as *intuitively clear* 'proofs' is that, different from the concept 'equality of area', the metrical concept of 'area' (of a rectangle) is anything but intuitively clear. Why the metrical concept of 'length' (of a line segment of the number line) meets no difficulties, is another question.

MATH. Whatever this may be, Klein's picture seems so convincing from an intuitive point of view, that it can be asked whether not only the sign rules, but also the fundamental properties of addition and multiplication, represented in formulas such as

$$a + b = b + a$$

and

$$ab = ba$$

should have similar perspicuous representations. This question will occupy us in in the rest of this hour, although I do not intend to argue that such representations would make proofs superfluous, as Klein suggested. Or did he only mean to say that they are sufficient for didactical purposes? For it is well-known that students for little affinity with 'abstract' thinking can be greatly helped by 'concrete' applications when they have to memorize or even 'understand' important theorems.

COMP. What do you precisely mean by 'abstract' and 'concrete'?

MATH. The standard explication of 'concrete' is 'in space and/or in time', and then 'abstract' is interpreted as 'neither in space, nor in time'. It follows that thinking is never abstract, but the idea is that looking at figures can make certain relationships more perspicuous. Considerations of this kind induced the Dutch mathematician Gerretsen to write an ‘Introduction to differential and integral calculus on an intuitive foundation’, but unfortunately his book never reached the status of a popular textbook. It is possible that its main title, ‘Raaklijn en oppervlakte’ – *Tangent and Area* – did not help to spread the book either. Yet its subtitle already indicates one of Gerretsen’s purposes: to provide ‘an insight into the scope and significance of fundamental mathematical concepts’ without bothering the reader with ‘detailed logical proofs’ in those cases in which ‘an intuitive argumentation’ suffices.[4]

Gerretsen’s approach lends itself admirably to the desired systematic treatment of perspicuous representations in algebra, but we shall see that an elaboration of one of his examples, the geometrical counterpart of the commutative property of the multiplication of real numbers, requires substantial geometrical knowledge. It follows that the ‘perspicuity’ of representation is a relative matter, not only depending on ‘mathematical intelligence’, but also on mathematical schooling. This aspect is easily neglected if one confines oneself to relatively simple examples such as Klein’s illustration.

LOG. The real number system can be introduced in several ways, and it is possible to present the theory without any reference to the number line, or even to any picture at all. I can imagine that some mathematicians prefer the ‘abstract’ presentation.

COMP. I am sympathetic to this view. Or, better, it seems that I have a personality type which loves algorithms. Perhaps that is why I became a computer scientist?

LOG. Hear! So why should we bother these ‘arithmetical’ personalities with pictures which might be a hindrance if the aim is mastery of the arithmetical rules?

MATH. It is only for elucidation of the arithmetical rules or laws that geometric constructions should be given. However some authors suggest that the former can be proved by the latter. This is not my point of view.

LOG. But what do you think of representations of relationships between numbers with the help of analytical geometry? I remember that Rosenbloom and Schuster showed that the commutative property of the

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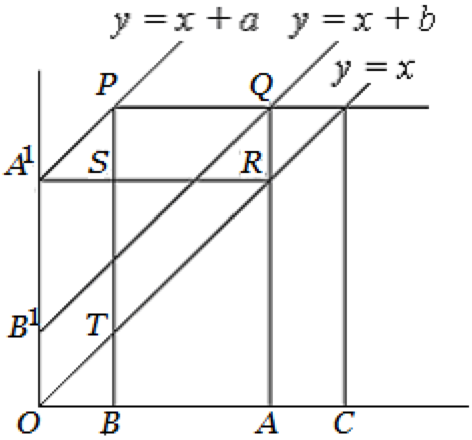
[4] J. C. H. Gerritsen, *Raaklijn en oppervlakte. Een inleiding to tde infinitesimaalrekening op aanschouwelijke grondslag*. Haarlem: Erven F. Bohn, 1959, p. VI.

addition: $a + b = b + a$ for positive real numbers can be represented in that way by a suitable two-dimensional picture.

They considered the graphs of $y = x + a$ and $y = x + b$ in order to determine the points P and Q such that $P = (b, b + a)$ and $Q = (a, a + b)$. Then one has only to show that the line through P and Q is parallel to the number line; in that case the projection on the number line of its intersection with the graph of $y = x$ is the desired unique point representing both $a + b$ and $b + a$. It is easily seen that $PQRS$, given that $R = (a, a)$ and $S = (b, a)$, is a rectangle because its vertical sides SP and RQ have the same length (b), albeit for different reasons:



($SP \cong A^1S$, $A^1S \cong OB$ and $RQ \cong OB^1$ and $OB^1 \cong OB$)

MATH. Is this really a perspicuous representation? Everything is in order, but somehow the 'evidence' that $a + b = b + a$ does not come from the figure 'on itself': one has first to 'move up' BP to the right until B coincides with C and then conclude that the length of OC is $b + a$ and subsequently to 'move up' AQ to the right until A also coincides with C and then conclude that the length of OC is also $a + b$ in order to 'see' that in fact $a + b = b + a$. Moreover this process is sustained by the acquired knowledge that PQ is parallel with SR and BA.

COMP. It follows that the notion of 'perspicuous representation' deserves a closer examination. What do you think, Math?

MATH. At this point we have the remarks by the philosopher who once tried to make this notion a cornerstone of his 'meta-philosophical' approach, Ludwig Wittgenstein. His functional characterization – that 'a perspicuous representation produces just that understanding which consists in *seeing connections*' – is useful because of its emphasis on ‘understanding’ in the indicated sense, although it neglects the proverbial ‘role of the subject’.

However, we have seen that Figure 4 does not give immediate insight in the equality of *a* + *b* and *b* + *a* since it includes the construction of the corresponding point on the number line.

LOG. It follows that its ‘perspicuity’ can be doubted. Is it better to pay attention only to *BP* which is first described as a combination of *BT* (with length *b*) and *TP* (with length *a*), and later is redescribed as a combination of *BS* (with length *a*) and *SP* (with length *b*)? Then the knowledge that the length of *SP* is *b* is still due to reasoning. But is this roundabout way necessary?

MATH. It is better to consult Gerretsen on this point. He demonstrated that there is a direct way of showing that the commutative property of the addition holds, using the configuration of the named points that we encountered in the line segment *BP* on the number line itself! Let us see how ‘perspicuous’ his solution is.

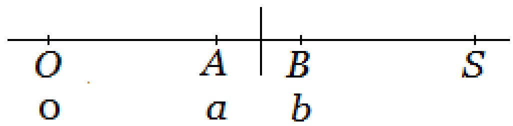


Figure 5

The idea behind this picture is that the number line is ‘moved up along itself’ until *O* coincides with *A*, and *B* with the indicated point *S*. Assuming that the points *O*, *A* and *B* are the counterparts of the numbers 0, *a* and *b*, we may conclude that *S* is the counterpart of *a* + *b*.

> ‘It is not difficult to see that the center of *AB* coincides with the center of *OS*. It follows that *A* and *B* fulfill the same role, with the result that the equality *a* + *b* = *b* + *a* strikes the eyes’.

LOG. Is this all that Gerritsen remarks? What does ‘fulfilling the same role’ mean in this case?

MATH. Apparantly some mathematicians may have sufficient support from a perspicuous picture in order to regard certain connections as obvious – *a word to the wise is enough* – whereas others are still puzzled by them, unless it is pointed out to them that if the number line is moved up along itself until *O* coincides with *B*, *A* will coincide with the point *S* as defined above, because of the symmetry of the figure. In any case, the perspicuous representation does not speak for itself. To borrow a dictum from Kant: ‘Intuitions without concepts are blind’ (*Anschauungen ohne Begriffe sind blind.*)

COMP. Did Kant not also say that thoughts without content are empty? (*Gedanken ohne Inhalt sind leer.*) But how would he then explain why computers can solve mathematical problems?

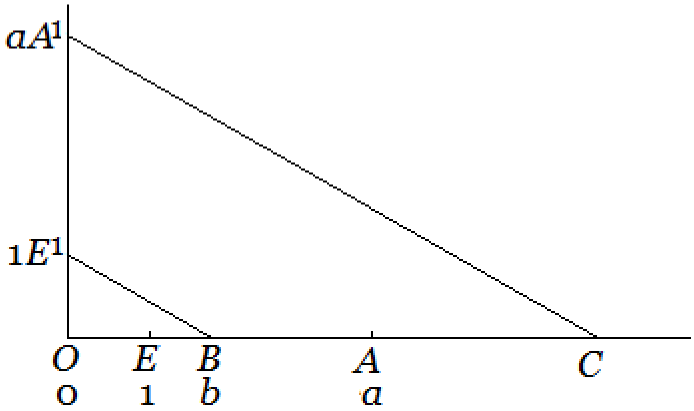
MATH. Well, they only solve problems because the programmers had mathematical thoughts which guided them in formulating and programming the algorithms. Take for example the number system itself. Of course, computers do not have thoughts about it, they ‘understand’ nothing about it in the sense that they can answer questions about its origin, extent, and purpose. Therefore the significance of the construction of the figures which show a number line must not be underestimated. The negative numbers can be introduced by it, postulating that each two numbers that lie on different sides of O on the number line, but with the same distance, are called ‘opposite’ numbers. If one of them is a, the other is written as $(-a)$ or simply as $-a$. It follows that $-(-a) = a$. Besides Gerretsen’s construction reveals that $a + (-a) = 0$.

COMP. I agree that a purely formalist teaching of negative numbers, without any reference to ‘external’ things, is meager, although it remains possible that students learn to correctly apply the arithmetical rules. I remember a paper by the Dutch mathematician Mannoury in which he recommended that students first learn the principal rules for complex numbers with the help of cards in the form of domino stones. But it is also true that they afterwards shopuld learn the fundamnental mathematical ideas behind complex numbers.

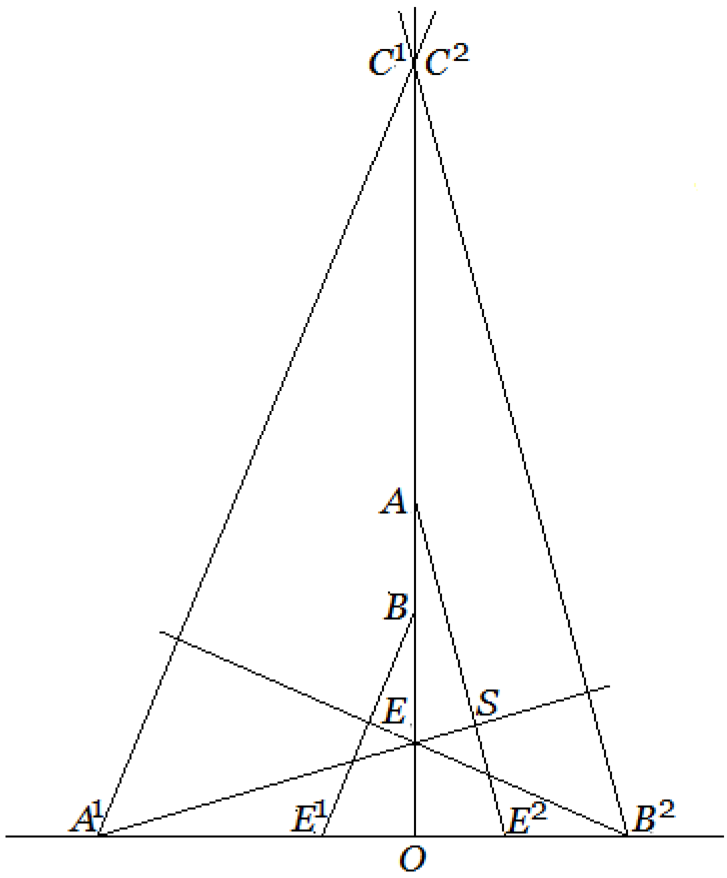
MATH. I am glad that you distinguish between matematical fundamentals and algorithms. Nevertheless I am aware of the fact that the ability of worrking with numbers is a prerequisite for a better understanding, and also that the drawing of conclusions from perspicuous representations may require a considerable amount of knowledge. This can be illustrated with the admittedly extreme but nice example of the commutative property of the multiplication. It is based on an idea for the construction of the product of two numbers by Gerritsen. Nevertheless the elaboration is new as far as I know.

LOG. That sounds interesting, so what is Gerritsen’s construction?

MATH. The geometrical construction of the product ab of the numbers a and b rests on the equation $1 : a = b : ab$ and therefore we do not only make use of the points A and B, but also of the point E, at a distance of 1 to O. In order to get similar triangles, A and E are rotated around O, preferably anti-clockwise over a right angle:



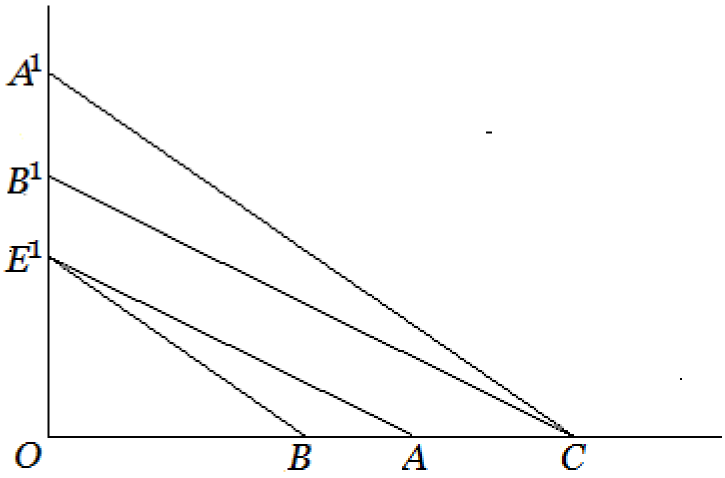
A^1C has been made parallel to E^1B, so the equation $OE^1 : OA^1 = OB : OC$ provides the condition that makes C the image of ab. In the same way we can determine the image of ba, but in order to get a perspicuous representation I also rotated B and E, albeit this time clockwise over a right angle around O. For a better visualization the number line has been drawn vertically:



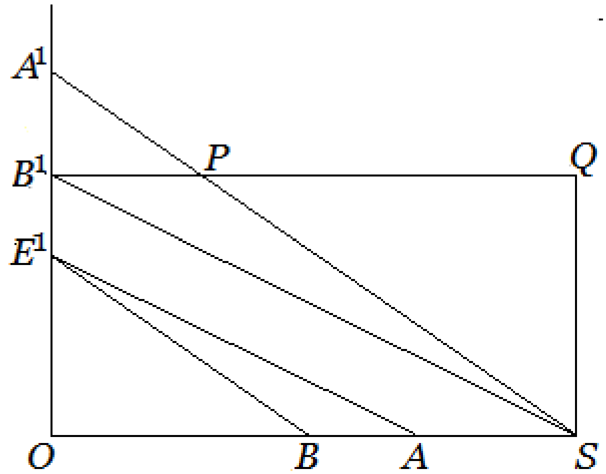
According to the construction, A^1C^1 is parallel to E^1B and B^2C^2 is parallel to E^2A. But there is more: since the triangles EOA^1 and E^2OA are congruent, the triangles A^1EO and AES are similar. It follows that $A^1S \perp E^2S$, hence that $A^1S \perp B^2C^2$ and $B^2C^2 \perp A^1E$. By analogy we conclude that $A^1C^1 \perp B^2E$. This means that A^1C^1, B^2C^2 and EO are the altitudes of the triangle A^1B^2E, and meet in one point! That is the reason why C^1 and C^2 coincide, and this means nothing else than that $ab = ba$.

LOG. Your picture can indeed be called a perspicuous representation of this algebraic theorem – after all the equality of *ab* and *ba* is directly visualized in the meeting of the product-determining lines in one and the same point of the number line. However without the geometrical knowledge that the three altitudes of a triangle meet in one point and without the insight that this knowledge is applicable in this case, one does not have the required mathematical evidence. I think that this finding is significant for any psychological theory of perspicuous representations: it is not only a misconception that perspicuous representations are "immediately evident", but it is also not tenable that they are ready to hand. There is a difference between genuine perspicuous representations of mathematical theorems and mere illustrations of them: the former can be accompanied by an explanation why the relationships hold in the figure, whereas in the case of the latter his cannot be done.

MATH. An example of nothing else than an illustration of a theorem is Gerretsen's own picture of the equality $ab = ba$, as long as it only appears from the drawn figure that the construction of the image of *ab* and that of *ba* produce the same point of the number line, for it is 'not immediately clear on which geometrical evidence this rests', as Gerretsen himself remarked:



LOG. Yet there is a simple geometrical proof of the theorem that the point of intersection S of the line through A^1 parallel with E^1B and the line through B^1 parallel with E^1A lies on the number line. To this purpose, we only have to draw the line through B^1 parallel to the number line – intersecting A^1S in P:



The proof that the distance of S to this line is equal to the length of OB^1, in other words that $SQ \cong OB^1$ can be done with the help of geometrical similarities, leading to proportionalities. In short:

$$\begin{aligned} SQ : A^1B^1 &= QP : B^1P \\ &= OB : AB \\ &= OB^1 : A^1B^1 \end{aligned}$$

The used equalities can easily be read off from the figure:

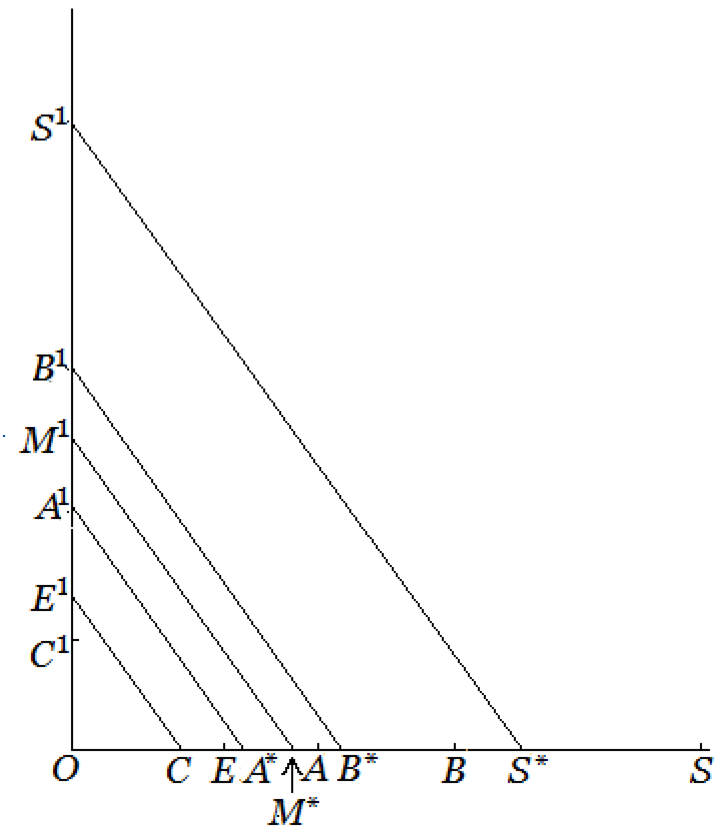
Once we know this, the question is whether thid picture yields sufficient 'geometrical evidence' for regarding it as a 'perspicuous representation' of the theorem that $ab = ba$. The fact that this is a meaningful question shows that the notion of 'perspicuous representation' is still vague.

MATH. But we can live with this vagueness as long as the representations, perspicuous or not, have manifest functions. As I noticed above, Wittgenstein emphasized that perspicuous representations help to see 'connections'. The example of my own picture shows that this can be true in an unexpected sense: it connected apparently unrelated theorems with each other, a result that is of vital importance for a subject that is related to my principal purpose, finding out principles and procedures of productive problem solving, namely the art of 'productive problem posing'. Figure 9 can have a similar function: it asks for an explanation of the connection of theorems about algebraic operations and theorems about geometrical similarities.

How close this connection is, should appear from the next example, the representation of the distributive law, $c(a + b) = ca + cb$. Following Gerretsen's example, we assume that $c > 0$.

Let A, B, and C be the images of a, b, and c. The points A^1, B^1, and C^1 are the result of a rotation around O. A^* and B^* are the image points of ca and cb, constructed with the help of the lines A^1A^* and B^1B^*, parallel to CE^1. The image S of $a + b$ is rotated into S^1. M^1 is the midpoint of OS^1, and also the midpoint of A^1B^1.

It follows that the line through M^1 parallel to CE^1 intersects the horizontal number line in the midpoint M^* of A^*B^*. Since M^* is also the midpoint of OS^* – with S^* as the image point of $c(a + b)$, constructed in the usual way with the help of a parallel line – the picture does indeed shows that $c(a + b) = ca + cb$.



But here we have apparently reached the limits of perspicuity. The picture does not show by visual means that S^* is also the image of $ca + cb$. This is merely concluded by reasoning, since the image of $ca + cb$ itself is not separately construed. Moreover Gerretsen did not even draw the points S and S^* in his own picture! In this respect, this picture is similar to Figure 4 – it may satisfy some mathematicians, but it may leave others with the feeling that they have missed something.

LOG. I have a question for Comp. Don’t you think that students who only get a training in algorithms will eventually desire to learn about the ideas which are behind them?

COMP. I am not sure. Students who only need mathematical rules in order to apply them in other disciplines, may well live with this ignorance. But talking for myself, I am glad that I got more background knowledge by Math’s explanations.

MATH. I am happy to hear that.