PENTRIES AND HEPTRIES, A SEQUEL TO SQUANGLES

LOG. (*as she meets Comp at the entrance of the L. E. J. Brouwer Institute*) Hallo
Comp, what's new?

COMP. Nothing special, except perhaps that I had a discussion with our colleague Math. You know that he has a low opinion of computer scientists such as me. This time he tried to convince me that mathematicians lean heavily on intuition.

LOG. What does he mean by that? I am always suspicious of arguments based on intuition. I don't even believe in female intuition. To me, intuitions are nothing else than primitive prejudices. They only seldom reveal the truth.

COMP. No, no, Math's account is subtler. He distinguished intuitive inferences from promising conclusions. Intuitive inferences may spontaneously arise in everyone and at several occasions, but mathematicians sometimes draw promising conclusions from them and in that case they may take heuristic advantage from their intuitions. That ability is of course dependent on someone's knowledge or problem solving abilities.

LOG. Can you give an example?

COMP. Well, that is just what Math did some days ago. He defined squangles as those natural numbers that are both a square and a triangular number, for example 36 and 1225. You see, 1225 is not only the square of 35, but also the half of 49 times 50. He managed to find a general rule for deriving such numbers, because he took advantage of his good guesses, as I would say. (*They arrive at Comp's room.*)

LOG. It sounds interesting. I wished I had heard the story from himself, but he seldom takes notice of me, obsessed as he is with his mathematical recreations.

MATH. (*coming out of the blue*) Good morning Log, hallo Comp, how do you do?

LOG. I'm fine, we were just talking about you and your mathematical puzzles. Comp told me that you amused yourself with uh… squangles. He said that you found a rule, but what is the use of such elementary Diophantine equations?

MATH. I'm not studying Diophantine equations for their own sake, but I try to get more grip on how mathematicians work. One of my findings is that intuitions play an important role, though I distinguish between intuitive inferences with promising conclusions and those that have no direct consequences for the problem solving process. The squangles problem served my purpose quite well, but you as a trained mathematician can understand that I did not stop there, and tried to find interesting generalizations. This is just the reason that I came to see Comp today, for I think he can help me.

COMP. Does this mean that you at last recognized the benefits of the computer?

MATH. In a way, yes, but don't forget that mathematicians in the past already leaned upon computations, then done by accurate human calculators. As you may know, in nineteenth century England these people were called computers. Nowadays it is obvious that artificial computers should do the work. And after our last discussion I began to realize that computers, or rather, computer scientists, may offer heuristic support to mathematicians. Let me explain.

LOG. Is it interesting enough for me to stay? Intuition is not my forte.

MATH. Please stay with us, it is possible that you too can be of help to me, or even that you will solve the problems that I encountered in the generalization of the squangles problem that I dismissed as too difficult for me. May I? (*Goes to the whiteboard in Comp's room.*) Look here, my original problem was to solve the Diophantine equation

$$m^2 = ½n(n + 1)$$

My first idea was to continue with larger powers of m, so I considered the following equation.

$$m^3 = 1/6n(n + 1)(n + 2)$$

or, in general,

$$m^k = \frac{n(n + 1)(n + 2) \cdots (n + k - 1)}{k!}$$

But unfortunately this was not a good idea, for I could not find any non-trivial solution of this equation for values of *k* larger than 2. I made several calculations with paper and pencil.

COMP. That is why you came to me.

MATH. No, no. I have the feeling that there are indeed no genuine solutions at all, but proving it might be as difficult as proving Fermat`s conjecture.

COMP. But why do you not try, it might make you as famous as Andrew Wiles!

MATH. I'm sorry, that is beyond me. Perhaps it is more in your line, Log. You are good in reasoning about complex structures, aren't you?

LOG. Well, the above formula is at least very familiar to me. But I understand that you tried another generalization?

MATH. Have you ever heard of pentagonal numbers?

COMP. Yes, I do. I learned such things from a book by Edna Kramer, *The Main Stream of Mathematics*. I remember very well that she tells a story about Sonja Kovalevsky and her daughter who is playing with marbles. Well, Foufi, that is her name, built triangular numbers, squares, pentagonal and hexagonal numbers, by forming regular figures.

LOG. The defining formulas of such numbers are easily found if one knows that they appear as sums of sequences. Just as triangular numbers are formed by the successive sums from the sequence 1, 2, 3, 4, and so on, and square numbers from 1, 3, 5, 7, and so on, pentagonal numbers can be derived from 1, 4, 7, 10, and so on, with a constant difference of 3, and heptagonal numbers from 1, 6, 11, 16, and so on, with a constant difference of 5. It follows that the formula for pentagonal numbers is

$$p(n) = \tfrac{1}{2}n(3n - 1)$$

and the formula for heptagonal numbers

$$h(n) = \tfrac{1}{2}n(5n - 3)$$

Now I see what Math's generalized problem amounts to: finding numbers that are both pentagonal and triangular, to begin with. This means that he wants to solve the equation

$$\tfrac{1}{2}m(3m - 1) = \tfrac{1}{2}n(n + 1)$$

Is that true, Math? For this and similar problems have already been solved with the theory for quadratic Diophantine equations.

MATH. Yes, I know, but, apart from the fact that this theory is not easily accessible to non-mathematicians, what I am interested in is not so much the solution of such problems, as well as the role of intuition in the search for solutions, especially when the problem solver is not acquainted with the theory of Diophantine equations and its applications. This forbids getting solutions from books or other media in so far as they are derived with the help of the theory. But it is allowed to use a computer for complicated computations. And therefore my question to Comp is if he can provide me with more solutions of the equation that Log just put on the whiteboard. I found only non-trivial ones, as you can see from my table:

n	$\tfrac{1}{2}m(3m - 1)$	$\tfrac{1}{2}n(n + 1)$
0	0	0
1	1	1
2	5	3
.	.	.
12		210
.	.	.
20	210	

This gives the following short table:

0, 0
1, 1
12, 20

COMP. I see, this is too small for your famous intuition… OK, there you go! (*Turns to his computer and starts working on it, while Math tells Log more about his work on Squangles. After a couple of minutes, Comp goes to the whiteboard and enlarges the last table with the following result:*)

0, 0
1, 1
12, 20
165, 285
2296, 3976
31977, 55385

COMP. Is this enough, Math? What does your intuition say?

MATH. This is not a question of intuition, since I do not immediately see a connection between the numbers. But my experience with squangles tells me that from the third line on, each coordinate might be a linear combination of the preceding two coordinates. Let us therefore divide 165 by 12, that gives almost 14 and if the rule is indeed analogous to the rule with squangles, then the third x-coordinate 12 must be 14 times the second x-coordinate 1 minus the first x-coordinate 0 minus a constant which obviously is 2. This would mean that the first rule is

$$x_{n+2} = 14x_{n+1} - x_n - 2$$

Let us check this with the next x-coordinate. 165 must be 14 times 12 minus 1 minus 2, that is correct. I think we are ready as far as the x-coordinates are concerned.

LOG. Shouldn't we check it with 2296?

MATH. I don't think that it is necessary, but if you insist, 14 times 165 makes 2310, 2310 minus 12 is 2298 and this is indeed 2 more than 2296. Or do you still want a check with 31977?

LOG. No, this is enough. It seems that the situation is similar to the one sketched by Russell in his book *Human Knowledge*. Russell's example was that of the sum of the first n cubic numbers:

$$1 , 1 + 8, 1 + 8 + 27, 1 + 8 + 27 + 64, \ldots$$

and I never forget his reaction: "Mathematical intuition is by no means infallible as regards such inductions, but in the case of good mathematicians it seems to be oftener right than wrong." OK, I am prepared to follow your line of thought and shall guess what the rule for the y-coordinates will be. Is it perhaps

$$y_{n+2} = 14y_{n+1} - y_n + 6 \text{ (?)}$$

COMP. You are right, I already found the same rules (*points at the monitor*). I used the method developed earlier for squangles.

LOG. But still there is the problem how to prove these rules. Even Math will admit this.

MATH. Yes, and I shall try to demonstrate how my method of analyzing solutions works in this case. Let us go back to our original equation, omitting the factor ½:

$$m(3m - 1) = n(n + 1)$$

and write down the results of the corresponding substitutions:

$m = 0$ and $n = 0$ gives $0 \cdot -1 = 0 \cdot 1$
$m = 1$ and $n = 1$ gives $1 \cdot 2 = 1 \cdot 2$
$m = 12$ and $n = 20$ gives $12 \cdot 35 = 20 \cdot 21$
$m = 165$ and $n = 285$ gives $165 \cdot 494 = 285 \cdot 286$

Let us rewrite the third equation in order to see how the equality could arise:

$$3 \cdot 4 \cdot 5 \cdot 7 = 4 \cdot 5 \cdot 7 \cdot 3$$

Aha! This is promising, and since 165 is equal to 11 times 15, we divide 285 by 15, which gives 19, and we divide 494 by 19, which gives 26, and that is enough for writing down the following equality:

$$11 \cdot 15 \cdot 19 \cdot 26 = 15 \cdot 19 \cdot 26 \cdot 11$$

But now we can also analyze the trivial solutions as follows:

$$0 \cdot 0 \cdot 0 \cdot 1 = 0 \cdot 0 \cdot 1 \cdot 0$$
$$1 \cdot 1 \cdot 1 \cdot 2 = 1 \cdot 1 \cdot 2 \cdot 1$$

LOG. What is the use of all this?

MATH. I want to find a rule for deriving each series from the preceding one, not from the two preceding ones, in order to prove the above rules. So let us look at the successive quadruples:

0, 0, 0, 1
1, 1, 1, 2

$$3, 4, 5, 7$$
$$11, 15, 19, 26$$

It is nice to see that we have again a relation between three successive components in equations of the following form:

$$z = 4y - x$$

but that is not very interesting. But don't you see that every two lines have following form:

$$a, b, c, d$$
$$d + b, d + 2b, d + 3b, 2d + 3b$$

COMP. It seems that you are right, but to me it comes as a surprise!

LOG. Is this an example of your intuition, Math? Yet I don't see any promising conclusion.

MATH. But I do. I think that it is not difficult to prove that the pentagonal number of the product of $d + b$ and $d + 2b$ is equal to the triangular number of the product of $d + 3b$ and $2d + b$. And then the proof of the rules with the coefficient 14 must be an easy job. Shall I try it here?

COMP. Maybe you can do it for yourself, and return afterwards. In the mean time Log and I can do some other work.

MATH. OK. (*Leaves Comp's room.*) (*Comp and Log search on Internet for publications about pentagonal triangular numbers.*)

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MATH. (*returning after a quarter of an hour*) Hallo friends! I was right that the proofs were easy. Do you want to see them?

LOG. Maybe you can sketch the proofs?

MATH. All right. Listen, first of all, I determined the condition which d and b must satisfy in order to produce a pentry. To that purpose, I reduced the following equation:

$$\mathrm{p}((d + b)(d + 2b)) = \mathrm{t}((d + 2b)(d + 3b))$$

which amounts to the same as

$$(d + b)(d + 2b)(3(d + b)(d + 2b) - 1) = (d + 2b)(d + 3b)((d + 2b)(d + 3b) + 1)$$

and I found the equation

$$4b(d^2 - 3b^2 - 1) = 0$$

LOG. And then you proved that this equation holds for every line of the sequence of quadruples, by using mathematical induction?

MATH. Yes, indeed. That $d^2 - 3b^2 = 1$ is obvious for the first line, so let us assume that

$$d_n^2 - 3b_n^2 = 1$$

Surprisingly, it appears that

$$d_{n+1}^2 - 3b_{n+1}^2$$

or

$$(2d_n + 3b_n)^2 - 3(d_n + 2b_n)^2$$

is just equal to

$$d_n^2 - 3b_n^2$$

COMP. That is interesting, for when you were absent, Log and I found in Eric Weinstein's *World of Mathematics* on the Internet that the Diophantine equation for pentagonal triangular numbers has the same coefficients for the squares as your condition. This equation is, namely,

$$x^2 - 3y^2 = -2$$

MATH. I agree that it is interesting, and it deserves a closer look. But I am not quite finished, for I must still prove that the recursive formulas are correct. That we have

$$z = 4y - x$$

is almost trivial if we apply the construction rule:

$$a, b, c, d$$

$$d + b,\ d + 2b,\ d + 3b,\ 2d + 3b$$
$$3d + 5b,\ 4d + 7b,\ 5d + 9b,\ 7d + 12b$$
$$11d + 19b,\ 15d + 26b,\ 19d + 33b,\ 726d + 45b$$

and the proofs of the recursive formulas with coefficient 14 are also easy, only here we must use our equation

$$d^2 - 3b^2 = 1$$

LOG. I wonder how general your procedure is, Math. Would it also apply to heptagonal triangular numbers?

MATH. I cannot believe that the structure that I found in my analysis of solutions – I will write the lines once more –

$$0{\cdot}0{\cdot}0{\cdot}1 = 0{\cdot}0{\cdot}1{\cdot}0$$
$$1{\cdot}1{\cdot}1{\cdot}2 = 1{\cdot}1{\cdot}2{\cdot}1$$
$$3{\cdot}4{\cdot}5{\cdot}7 = 4{\cdot}5{\cdot}7{\cdot}3$$
$$11{\cdot}15{\cdot}19{\cdot}26 = 15{\cdot}19{\cdot}26{\cdot}11$$

is only reserved for pentries, so we need some heptagonal triangular numbers, or, as I would say, heptries, in order to see if they show the same structure. Their determining equation is, of course,

$$\tfrac{1}{2}m(5m - 3) = \tfrac{1}{2}n(n + 1)$$

The problem is again that we have to find a few solutions. I immediately see $m = 5$ and $n = 10$ and this outcome gives the following line:

$$1{\cdot}5{\cdot}2{\cdot}11 = 5{\cdot}2{\cdot}11{\cdot}1$$

but that is not enough. What do you think, Comp?

COMP. I think I can find some more solutions with the help of my computer.

LOG. Wait a moment! I would suggest that we shorten the process of finding solutions by construing a parameter representation for this equation.

COMP. What do you mean by that? Or do I need higher mathematics to understand it?

LOG. Not at all, listen, I add the following equation:

$$m/n = p/q$$

and solve the following system of two equations:

$$m(5m - 3) = n(n + 1)$$
$$mq = pn$$

starting with

$$pm(5m - 3) = pn(n + 1)$$

or

$$pm(5m - 3) = mq(n + 1)$$

or

$$p(5m - 3) = q(n + 1)$$

and we have the two equations

$$5pm - qn = 3p + q$$
$$qm - pn = 0$$

COMP. I see, now we can express m and n in terms of p and q. Shall I continue?

LOG. Go ahead.

COMP. (*after some elaborations*) Here you are:

$$m = p(3p + q)/ (5p^2 - q^2)$$
$$n = q(3p + q)/ (5p^2 - q^2)$$

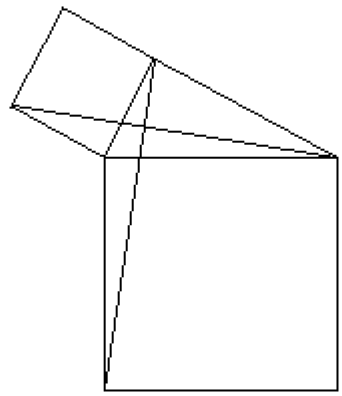
MATH. I think that you, Comp, can now help us by finding values of p and q such that the denominator $5p^2 - q^2$ of these fractions is small enough, preferably 1 or 4, in order to get whole numbers as outcomes.

COMP. Why not 2 and 3?

MATH. It is easy to see that there are no whole numbers as values of p and q such that $5p^2 - q^2$ is equal to 2 or 3.

LOG. Yes, but it seems also wise to remark that it suffices to take only values of q that end in a two or an eight for the equation $q^2 + 1 = 5p^2$ and only values of p that end in a one or a nine for $q^2 + 4 = 5p^2$.

COMP. We will see. (*Comp uses his computer, whereas Math and Log have a discussion about intuitive inferences that are not followed by promising conclusions. One of Math's examples is derived from Euclid's proof of the Pythagorean Theorem. In this proof, two lines are drawn in order to form congruent triangles – as shown in the following Figure. Then it is noticed that these lines are perpendicular. However it is by no means clear what this might contribute to a proof. The intuitive inference is correct, but the promising conclusion is absent.*)



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COMP. I have some numbers for you, Math:

$$h(5) = t(10)$$
$$h(221) = t(493)$$
$$h(1513) = t(3382)$$
$$h(71065) = t(158905)$$

but then my computer gave up.

MATH. Thank you, Comp. This is enough. I will now first write down the products:

$$5 \cdot 22 = 10 \cdot 11$$
$$221 \cdot 1102 = 493 \cdot 494$$
$$1513 \cdot 7562 = 3382 \cdot 3383$$
$$71065 \cdot 355322 = 158905 \cdot 158906$$

and then find the suitable factors:

$$1\cdot5\cdot2\cdot11 = 5\cdot2\cdot11\cdot1$$
$$13\cdot17\cdot29\cdot38 = 17\cdot29\cdot38\cdot13$$
$$17\cdot89\cdot38\cdot199 = 89\cdot38\cdot199\cdot17$$

and (*after some scribbling in a corner of the whiteboard*)

$$233\cdot305\cdot521\cdot682 = 305\cdot521\cdot682\cdot233$$

LOG. That is strange, the numbers of the first and the third line are not in order of magnitude. It follows that there is no recursive rule connecting all of them at once.

MATH. Apparently there are two sequences, so let us make two separate columns:

1, 5, 2, 11	13, 17, 29, 38
17, 89, 38, 199	233, 305, 521, 682

It is obvious that each of them satisfies the recursive rule $z = 18y - x$, so we only have to add a first line in both cases:

1, 1, -2, -1	1, 1, 1, 2
1, 5, 2, 11	13, 17, 29, 38
17, 89, 38, 199	233, 305, 521, 682

More difficult is the question how to find a direct connection between the components of two successive lines, a *bd*-rule of the same kind as we used with the pentries. For the time being I see only the following relationship:

a	b	c	d
x	y	z	$20b + 9d$

LOG. That is your intuition! But what is your promising conclusion?

MATH. That there are similar dependencies between the other components, and that it is fruitful to look for them, I think by means of linear equations.

COMP. So far as I can see, $z = 20a + 9c$, isn't it?

MATH. Excellent! And this means that we can find the other relationships by an application of the 18 rule:

$$\begin{array}{cccc} a & b & c & d \\ x & y & 20a + 9c & 20b + 9d \end{array}$$

$$18(20a + 9c) - c =$$
$$20x + 9(20a + 9c)$$

COMP. You know that I am good in solving linear equations! (*Comp works the last equation out.*)

COMP. See here:

$$x = 9a + 4c$$

And, no doubt,

$$y = 9b + d$$

LOG. I agree. Now we can use the same procedure as in the case of the pentries, in order to prove the recursive rule. First, we determine the condition for

IF $h(ab) = t(bc)$, THEN $h((9a + 4c)(9b + 4d)) = t((9b + 4d)(20a + 9c))$

Then we prove this condition by mathematical induction, and finally we prove the recursive rule.

MATH. That's it. Once more we have seen that mathematical problem solving is a complex activity with intuitive inferences, promising conclusions, calculations and other applications of known rules.

LOG. But all these activities are, in a sense, steered by a more or less conscious logical process in which reasoning is central, not only deductive and inductive, but also abductive. By abductive reasoning I mean the kind of reasoning in which possible premises are formulated for given conclusions. It would be interesting to determine, from a logical point of view, the structure of Math's actual problem solving process, his intuitive inferences included. I think that the possibility of such inferences may also depend on certain attitudes: Math was searching for something, trying to find something, but what was it that he was looking for? We might learn a lot from such an analysis of Math's thought process. If only we had recorded our discussion of pentries en heptries...

COMP. (*with a smile*) Who knows!

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