NATURAL VERSUS ARTIFICIAL PRODUCTIVE PROBLEM SOLVING

(*Comp, Log and Math have come together in Math's room. After some small talk, Log tries to start a serious discussion.*)

LOG. We have already had many discussions about 'productive problem solving', Math, and we have seen several examples of different solutions, given either by a computer or by a mathematician. Nevertheless I still miss a philosophical distinction, if not definition, of the two ways of problem solving, given that both can be productive.

COMP. I am glad that you acknowledged that computer solutions can be productive, Log. It improves on the rather unfruitful terminological discussions about the intelligence of artificial intelligence, so to speak.

MATH. I agree, but there are differences, as Log rightly remarked, although I do not want to go into a philosophical discussion of them.

COMP. I do not speak for Log, but I am an anti-philosopher just like you.

LOG. You know that I worked at the E. W. Beth Institute for Foundations of the Exact Sciences, and this institute was a section of the Philosophy Department before it went over to the Science Department. That is the reason why I still use the word 'philosophy' for discussions *about* mathematics, in other words for dealing with meta-problems.

COMP. I have no difficulties with it, as long as it implies no *Gefasel*, I mean: waffle.

LOG. What do you take me for?

COMP. I always appreciate your philosophical remarks, LOG.

MATH. So do I, and Log's philosophical question about differences in natural and artificial productive problem solving can not be dismissed as unproductive. Moreover I thought about this myself last week and I have at least some concrete examples at hand with which the differences can be illustrated.

COMP: Perhaps we can draw some conclusions about them and give a preliminary answer to Log's question.

LOG. I hope so, go ahead, Math.

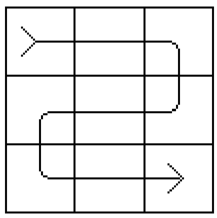
MATH. All examples concern cases in which a state space problem has no solution at all. This result can be or has been established by a computer program, simply by searching the whole space, but human problem solvers reached the same conclusion in a different way. Perhaps this is important for Log's meta-problem. But let me begin with my first problem. You know the eight puzzle with its goal state

1	2	3
8		4
7	6	5

You know also that half of the eight puzzle problems have no solution, with the following state as a characteristic example:

8	1	2
7		3
6	5	4

The Dutch mathematician/linguist/writer Hugo Brandt Corstius gave a simple explanation for the unsolvability states with the fifteen puzzle.[1] He introduced a 'snake ordering' that we can also apply in the eight puzzle:



He noticed that moving a tile downwards to the empty space, for example the 1 in the above characteristic example, can be described as a 'jump' over an even number of tiles, and similarly moving a tile upwards as a jump 'back', also over an even number of tiles, and a horizontal move as a jump over zero tiles, an even number as well. It follows that it is impossible to interchange two tiles and keep the other tiles in the same

[1] Battus. *Vrij Nederland*, 12.07.1980.

place. For two adjacent tiles to interchange their position, one of them would have to jump over only one tile.

COMP. I don’t see it.

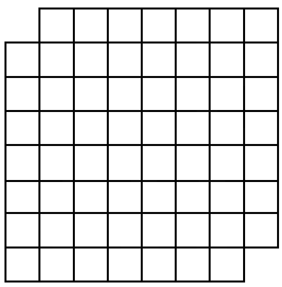
MATH. Suppose we could interchange the 1 and the 2. Then we get the following simple solvable problem:

8	2	1
7		3
6	5	4

COMP. I see, but we cannot reach this state, so the original problem has no solution.

MATH. But suppose that a computer program that can perform all series of elementary moves, decides that the end state cannot be reached when all possible states have been passed. This program is productive, but this is achieved in a quite different way from Hugo’s solution.

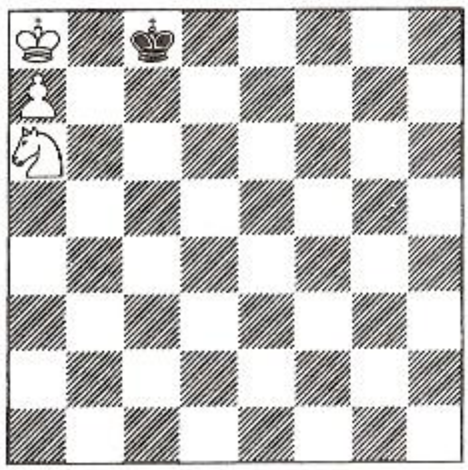
COMP. This reminds me of the famous tiler problem, to cover an area of 8 by 8 of which two opposite corners of 1 by 1 are lacking, with tiles of 2 by 1:



MATH. It is my favourite example of a problem solution with the help of an anti-abstract model. Colouring the tiles black and white

means introducing differences that are not present in the original. Again, a computer program which discovers that the problem is unsolvable after it has gone through all possibilities, gives a completely different solution than the person who models the area as a mutilated chess board.[2]

LOG. Talking about chess, I remember a chess problem by Lasker[3] in which White to move cannot win, because his knight will always cover a field of the wrong colour.



LOG. Until now, we haven't seen logical or mathematical problems. I'd like to see how they are solved differently by computers and men.

MATH. I will give an example of a problem that can arise when we consider the following axioms in finite geometry:

(1) There are exactly 10 points
(2) There are exactly 5 lines
(3) Each line contains exactly 4 points
(4) Every two lines have exactly 1 point in common

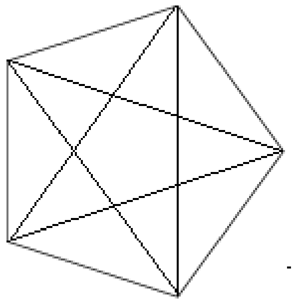
LOG. Is this a joke? What a strange axiom system! Where did you find it?

MATH. Wait until you have drawn a model of this system!

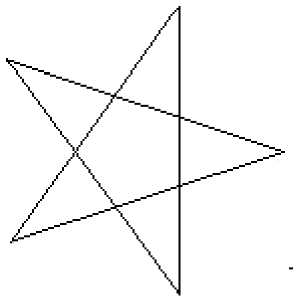
LOG. Let me think. Intuitively, the 10 points and 5 lines suggest that it has something to do with pentagons or decagons. (*She draws the following figure*)

---

[2] Cf. Philip C. Jackson, Jr., *Introduction to Artificial Intelligence*. New York: Petrocelli Books, 1974, p.113.
[3] Emanuel Lasker, *Lehrbuch des Schachspiels*. Berlin: Wetbuchhandel, 1926, p. 21.



Ah, here is your model:



MATH. Excellent! Now for the problem. Can you put numbers at the points in such a way that the sum of every four numbers on the same line is everywhere the same?

COMP. There are five such sums, but each number occurs twice, so the sum is one fifth of the double of the sum of the numbers 1 to 10.

LOG. Yes, it is 22. Given the preceding problems, I assume that the problem has no solution, but how do we prove this?

COMP. I will write a computer program. I will be back in a few minutes. (*He leaves the room.*)

MATH. In the mean time I will sketch a non-artificial proof. Let me first write all possible sums containing the number 1:

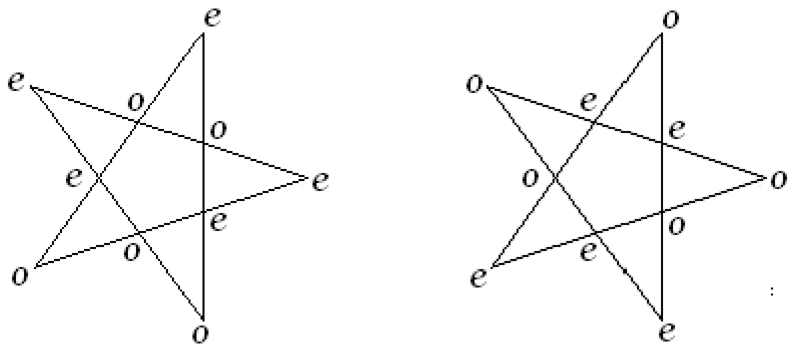
1 + 2 + 9 + 10
1 + 3 + 8 + 10
1 + 4 + 7 + 10
1 + 4 + 8 + 9
1 + 5 + 6 + 10
1 + 5 + 7 + 9

Apparently each sum consists of two odd and two even numbers, with the exception 1 + 5 + 7 + 9 with only odd numbers.

LOG. Clearly the other exception is 2 + 4 + 6 + 10 with only even numbers. But these two sums are useless. They swallow almost all numbers of the same kind. If there is a solution, then each sum always consists of two odd and two even numbers. This has consequences for the general shapes of a solution, but that is relatively unimportant.

LOG. Nevertheless I would like to see them.

MATH. Look:

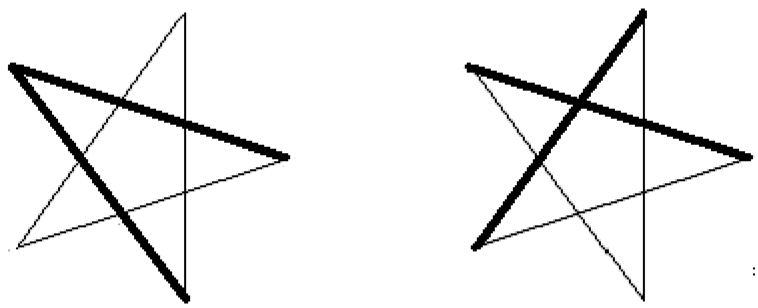


LOG. Nice, but it is obvious that the one can be derived from the other, simply by replacing each number n by its complement $11 - n$ But go on.

MATH. I return to my short list, omitting the last sum:

1 + 2 + 9 + 10
1 + 3 + 8 + 10
1 + 4 + 7 + 10
1 + 4 + 8 + 9
1 + 5 + 6 + 10

Now I need some geometry. First of all, each point lies on exactly two lines. This means that we have to select two of the above sums. Secondly, each two lines have exactly one point in common. This excludes the first three sums. and the last two, 1 + 4 + 8 + 9 and 1 + 5 + 6 + 10 must be chosen. But look now at the following figure:



These are the two different possibilities for the position of these two lines. In both cases there are three places left, and they have to be filled by the numbers that are not yet used: 2, 3, and 7. However, because the open places form a triangle, they cannot all three lie on the same line. But each two of them do. The only remaining line containing 2 and 3 is represented by the sum 2 + 3 + 8 + 9. But this line has two points in common with the line represented by 1 + 4 + 8 + 9, and that is excluded. There is no way out any more. The problem has no solution at all.

LOG. I am completely convinced.

COMP. (*returning*) There is definitely no solution, and I assume that you came to the same conclusion in your way! My program checked all possibilities. But the blackboard shows that you had to do some systematic work as well.

MATH. That is true. But the difference with a systematical search through the whole state space is obvious. There was ample attention to structural properties which a solution, if any, would have.

LOG. So your presentation of this Magical Star problem by means of a geometrical axiom system helped you, and the same holds for the closer inspection of the sums.

MATH. Indeed, so instead of distinguishing 'form' and 'content' as some philosophers do, in order to argue that computers can only do 'formal' work, I think that mathematicians can regard mathematical problems as related to other problems, and thereby find other ways of solving them.

COMP. Do you mean that associations are important? I thought that the time is over that psychologists put all their cards on associations.

MATH. There is nothing wrong with associations, as long as one does not think that there are laws about them. What matters is that trained mathematicians have seen and given all sorts of solutions. Faced with a new poblem, they sometimes think of one of them, maybe only in the form of an intuitive inference together with a promising conclusion to the effect that it is worth while to work it out for the present problem. I admit that this is very vague, but I know no better way of articulating this idea. Psychological analysis of mathematical activities is still in its infancy, but this is a difference with computer programs: talking about their psychological processes is nonsense.

LOG. What do you think of the thought processes of computer scientists?

MATH. My remarks have only to do with mathematical abilities, about computer scientists I hesitate to give an opinion, although they are also mathematicians of a kind.

COMP. Hm. Let us go back to the problem. I have not seen your derivation, Math, but can you conclude from it about the fundamental difference between the computer proof and yours?

LOG. I would suggest that the computer only showed *that* there is no solution, whereas you showed *why* there is no solution. This can be easily generalized. Am I right, Math?

MATH. Superficially, it looks as if this is the criterion we need. In the examples of the eight puzzle and the tiler problem, there was a short answer why a certain configuration had no solution. But as is mostly the case with why-questions, the five point star problem has no simple answer to the question why it is unsolvable. And since we are not philosophers who are satisfied with simple slogans, we must dismiss why-questions as irrelevant.

LOG. But don't you want an *explanation* of the fact that a certain problem has no solution?

MATH. The proof is the explanation!

COMP. Does this mean that the computer proof in which all possibilities are gone through is also an explanation?

MATH. Yes, why not? The difference with my explanation, I mean, proof, is only that mine is more perspicuous, at least for us. Moreover it does not stand alone, like the other solutions. Hugo's approach to a fifteen problem position could without difficulties be taken over for an eight problem position, the solution of the tiler problem can also be attempted for similar problems. Lasker's problem seems to have a general character, because two moves were sufficient to generate the impossibility. And I have the feeling that my way of dealing with the pentagon problem might be used in other problems.

COMP. Does this not also hold for the computer way of checking all possibilities?

MATH. In a sense it does, but it has taught us nothing new, apart from the result. Moreover, even a slight change in a problem situation might again require a full systematic search through all possibilities, whereas human solution procedures and strategies might immediately be applied without much search or even without any serious search, that is, after a brief inspection of the situation only.

LOG. Is this then a significant difference between productive problem solving procedures by men and by computers?

MATH. It is a difference, but I think that it is not very fruitful to call it significant. What does that mean? And should such a difference hold for all times, as philosophers want it to be? That would be preposterous. Moreover I do not exclude that computer programs will use more and more humanlike procedures, whereas I know as well as you that human beings sometimes resort to computerlike procedures, although these are also invented by human beings.

COMP. But only after the invention of computers!

MATH. You are right, Comp. But what I want to stress is that mathematicians should strive for solutions that are as perspicuous as possible. In this respect, my proof of the impossibility theorem for a five point magical star is perhaps not the last word. On the other hand, computer scientists should exploit the best possibilities of computers. This time we discussed only examples of problems that have no solution, but you know as well as I that it can be very difficult for human beings to give all solutions of solvable problems. I know from experience that we can sometimes find special solutions without great effort, but also that it can be very difficult to sum up, let alone to specify all solutions. In that case we can go to the computer scientists with their systematic procedures.

LOG. There is still a difficulty with your Magical Star problem, Math. Can it be modified in such a way that it does have a solution?

MATH. It can, as I saw in a Dutch book, written by the mathematician Stutvoet.[4] He omitted the 10, and added the condition that the 5 appears twice. Solutions can easily be found. In order to facilitate the search process, I give the second 5 an underscore. Then there are five sums, or lines, containing the 1:

1 + 2 + 8 + 9
1 + 3 + 7 + 9
1 + 4 + 6 + 9

[4] Stutvoet (1942), p. 186 and 217-218.

1 + 4 + 7 + 8
1 + 5 + 5 + 9
1 + 5 + 6 + 8

We can omit 1 + 2 + 8 + 9 and 1 + 4 + 6 + 9, because they have two points in common with each of the other lines, and 1 + 5 + 5 + 9 falls off because it contains no even numbers.

~~1 + 2 + 8 + 9~~
1 + 3 + 7 + 9
~~1 + 4 + 6 + 9~~
1 + 4 + 7 + 8
1 + 5 + 6 + 8
~~1 + 5 + 5 + 9~~
1 + 5 + 6 + 8

Two lines remain:

1 + 3 + 7 + 9
1 + 5 + 6 + 8

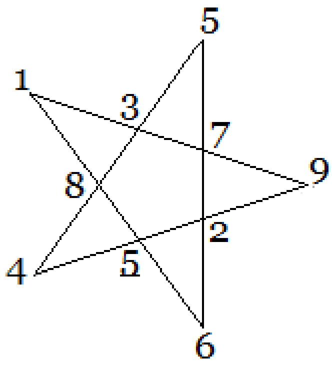
Proceeding with lines containing the 2, we get

~~2 + 3 + 6 + 9~~
~~2 + 3 + 7 + 8~~
2 + 4 + 5 + 9
2 + 5 + 6 + 7

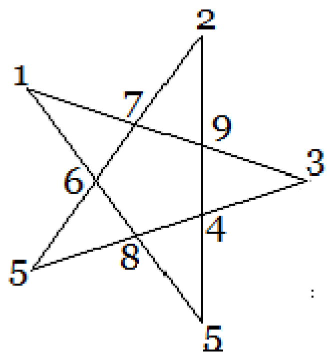
The complete solution requires only one more line:

1 + 3 + 7 + 9
1 + 5 + 6 + 8
2 + 4 + 5 + 9
2 + 5 + 6 + 7
3 + 4 + 5 + 8

When we keep the order of 1 + 3 + 7 + 9 in the figure, we get:



But this is not necessary. Here is another, but similar solution:



LOG. This problem is too easy, I think.

MATH. That is not the reason why I mentioned it. The question is how the modification with the two fives was found! I don't know the answer, but I am quite sure that this would be an insoluble problem for a computer.

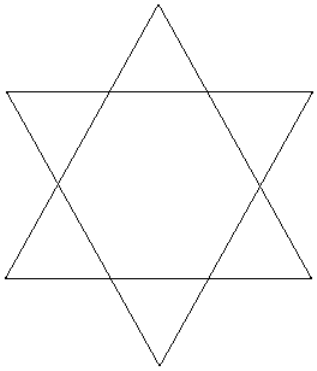
COMP. Hm. I suggest that we are going to investigate the solutions for larger Magical Star problems, to begin with the Mageen David!

LOG. How do you know that this star has solutions?

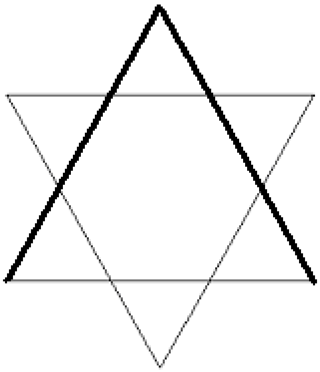
COMP. My former mathematics teacher gave it once as an exercise!

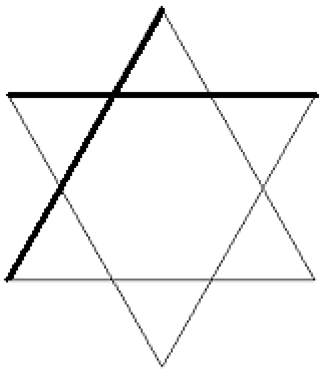
MATH. (*laughing*) I know.

(*Hereby their discussion ends and Log and Comp leave Math's room whereas Math's thoughts go back to the time he propounded the problem of the six-pointed star to his students. He wonders if it would be possible to find a solution with a procedure similar to the one he followed for the five-pointed star, with 12 points and sums 26. How he proceeded is reproduced below.*)



All lines containing 1 can be listed as follows:





1 2 11 12
1 3 10 12
1 4 9 12
1 4 10 11
1 5 8 12
1 5 9 11
1 6 7 12
1 6 8 11
1 6 9 10

There are exactly two lines containing 1. A line containing 1 can only have this point in common with the other line containing 1. This reduces the possiblities of such lines:

1 2 11 12 can only go together with 1 6 9 10
1 3 10 12 only together with 1 5 9 11 and 1 6 8 11
1 4 10 11 only together with 1 5 8 12, 1 6 7 12, and 1 6 9 10
1 5 8 12 only together with 1 4 10 11 and 1 6 9 10
1 5 9 11 only together with 1 3 10 12 and 1 6 7 12
1 6 7 12 only together with 1 4 10 11 and 1 5 9 11
1 6 8 11 only together with 1 3 10 11
1 6 9 10 only together with 1 2 11 12, 1 4 10 11, 1 5 8 12

(1) Let us begin with the first two lines, 1 2 11 12 and 1 6 9 10.

1 2 11 12 is only “parallel” with

3 4 9 10
4 5 7 10
4 5 8 9
4 6 7 9

Inspection of these lines shows that only 4 5 7 10 and 4 5 8 9 have exactly one point in common with 1 6 9 10.

1 6 9 10 is only “parallel” with

2 4 8 12
2 5 7 12

Both parallels are useless, they have two points in common with 1 2 11 12.

(2) We take the next two lines, 1 3 10 12 and 1 5 9 11.

1 3 10 12 is only “parallel” with

2 4 9 11 useless (two points in common with 1 5 9 11)
2 5 8 11 useless (two points in common with 1 5 9 10)
2 6 7 11

1 5 9 11 is parallel with 2 4 8 12

Thus we selected four lines:

1 3 10 12,
1 5 9 11,
2 6 7 11, and
2 4 8 12.

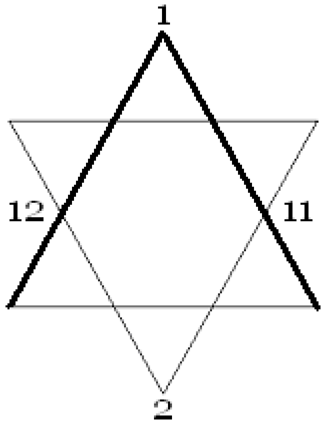
We only need two parallel lines for completing the figure. We can select them from the remaining lines:

3 4 9 10	4 5 7 10	5 6 7 8
3 5 8 10	4 5 8 9	
3 6 7 10	4 6 7 9	
3 6 8 9,		

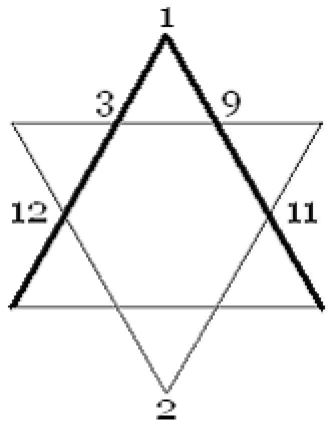
after the lines are eliminated which have two points in common with the selected lines. It appears that there are just two lines left:

3 6 8 9 and 4 5 7 10

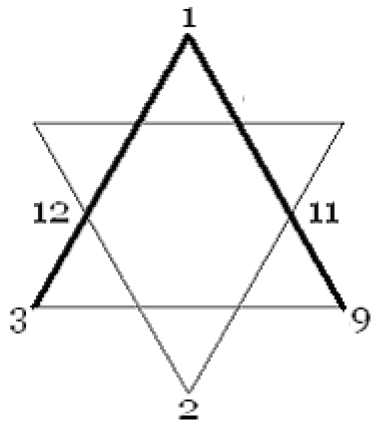
Assuming that the 1 is placed at the top of the figure, and the line 1 2 11 12 runs to the left behind, it is clear that the points 11, 12, and 2 must be placed outside the last two parallel lines, as follows:



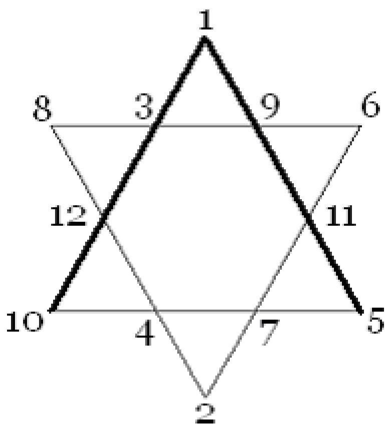
There are two possibilities for the intersection points of 3 6 8 9 and the first two lines 1 3 10 12 and 1 5 9 11:



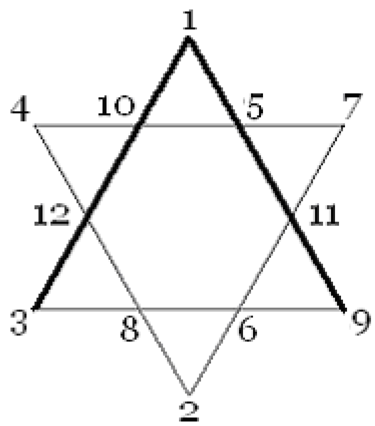
and



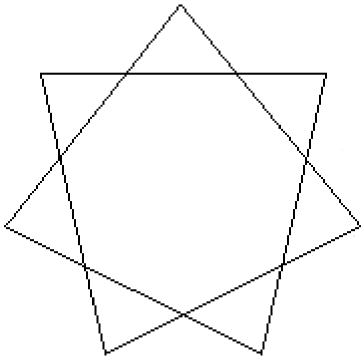
Both easily lead to a solution:



and



(*Hereafter Math solved the problem of the seven-pointed star, with 14 points and sums 30, in the following way.)*

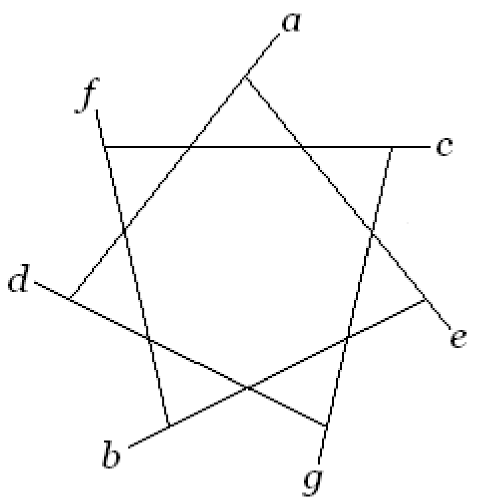


Let me first list all lines with sum 30:

1 2 13 14	2 3 11 14	3 4 9 14	4 5 7 14	5 6 7 12	6 7 8 9
1 3 12 14	2 3 12 13	3 4 10 13	4 5 8 13	5 6 8 11	
1 4 11 14	2 4 10 14	3 4 11 12	4 5 9 12	5 6 9 10	
1 5 10 14	2 4 11 13	3 5 8 14	4 5 10 11	5 7 8 10	
1 5 11 13	2 5 9 14	3 5 9 13	4 6 7 13		
1 6 9 14	2 5 10 13	3 5 10 12	4 6 8 12		
1 6 10 13	2 5 11 12	3 6 7 14	4 6 9 11		
1 6 11 12	2 6 8 14	3 6 8 13	4 7 8 11		
1 7 8 14	2 6 9 13	3 6 9 12	4 7 9 10		
1 7 9 13	2 6 10 12	3 6 10 11			
1 7 10 12	2 7 8 13	3 7 8 12			
1 8 9 12	2 7 9 12	3 7 9 11			
1 8 10 11	2 8 9 11	3 8 9 10			

After an attempt to follow a procedure similar to the one used in the solution of the six-pointed star, a closer inspection of the figure resulted in a happy idea: start with one line, say a, and find the next line, b, in the list which has no point in common with a. Then find the next line, c, which has no point in common with b, but exactly one point in common with a. Then find the next line, d, which has no point in common with c, but exactly one point in common with the lines preceding c. Repeat this procdure in order to find the lines e, f, and g, with the exception that the last line, g, has not only no point in common with f, but als no point in common with a:

$$a \to b \to c \to d \to e \to f \to g \to a$$



Starting with the first line for *a*:

a 1 2 13 14, the first possibility for *b* is
b 3 4 11 12, but then there is no *c* available any more.

Therefore the second possibility for *b* is chosen:

a 1 2 13 14
b 3 5 10 12, and then the procedure goes without breaks:
c 2 8 9 11
d 3 6 7 14
e 4 5 8 13
f 1 6 11 12, and now only the points 4, 7, 9, and 10 are still available, so:
g 4 7 9 10, just what is needed.

By considering the intersection points, starting with *a* and *c*, *a* and *d*, and so on, until *e* and *g*, the complete solution is found: