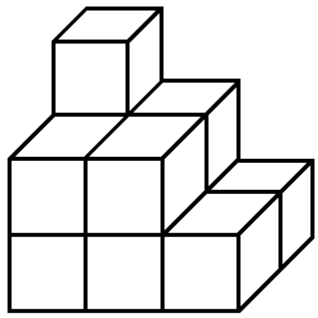
MEREOLOGICAL INFERENCES

MATH. When I was young, I sometimes amused myself by drawing so-called droodles, small and simple sketches, involving the task of guessing what they represent. Probably children will always be intrigued by such puzzles and curious about the solution. I was also fascinated by puzzles in which several blocks were depicted and their exact number had to be determined. Wait, I have an example at hand (*He shows the following picture*):



LOG. For us it is simple. How do computers solve such problems, Comp?

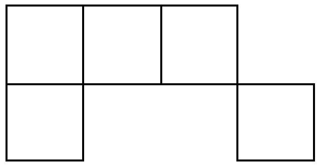
COMP. Humph! What is your point, Math, bringing in problems which are at most suitable for psychological tests?

MATH. Superficially, such puzzles are only a pastime. Yet they have a common property which deserves closer attention, namely, that only parts of an intended situation are represented in the picture. In order to find or to understand the solution, one has to regard certain elements as parts of larger wholes and also imagine that certain wholes have parts that are not depicted. In other words, part-whole inferences and whole-part inferences are needed.

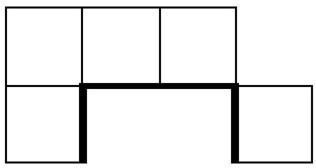
I am going to argue that 'mereological inferences', as I will call such inferences, form an important problem solving procedure in mathematics. One wonders why the psychological and instructional mathematical literature pays so little attention to them. Did psychologists, for example, notice that these inferences occur in the matchstick problems that sometimes occur in intelligence tests?

LOG. I have a strong impression that test psychologists do nothing more than repeating the traditional tests, and derive hypotheses about the relative difficulty of such a test from the response statistics. Are you going to confront us with an awkward matchstick problem, Math, in order to test our intelligence?

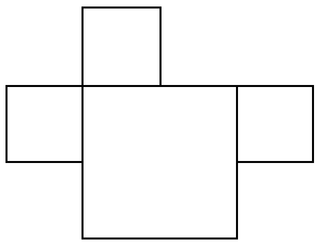
MATH. Not at all. My matchstick problems serve only as an illustration. Consider the following clear case. Suppose that we are asked to replace *four* matches in the following configuration of five squares in such a way that four squares are produced (*he uses the blackboard*):



The trick is that one of the ‘new’ squares must be four times as large as the old squares, and when this has been noticed, the task is to see that the figure already contains a part of such a square. An example is indicated in the following figure (*He broadens part of the picture*):



As soon as this is seen, a solution is easily found (*he makes some changes in the picture)*:



However, the simplicity of this example must not blind us to the fact that the problem is not yet *solved* by a particular mereological inference. It is not implied that exactly four matches are needed for the required goal situation.

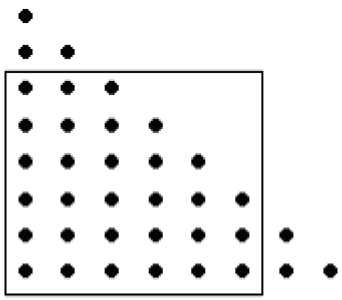
LOG. The original picture indeed contains more parts of larger squares than you have used in your solution; it even contains parts of squares that are nine or sixteen times as large as the original squares. I think that if necessary the problem solver may need calculations in order to determine whether there is a solution with such squares.

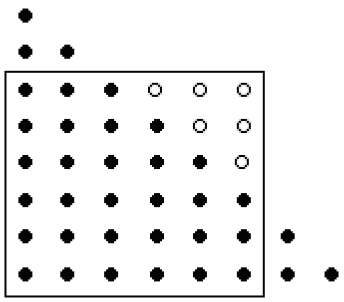
MATH. This is an important remark. It demonstrates that you can have an ‘intuitive insight’ without Helmholtz’s ‘conviction of correctness’. Remember that I introduced the term ‘promising conclusions’ in my discussion of intuitve inferences.

LOG. This example also shows that there need not be any question of ‘intuition’ at all. As soon as a solution is shown to a problem solver, he may start on a systematic search for all the possibilities to form larger squares.

COMP. This is the way in which computers should solve such problems, I think.

MATH. Perhaps match stick problems are not good examples for the proposition that mereological inferences are sometimes indispensable for solving a particular problem. Let me therefore present some examples which we discussed earlier. We have seen that the first approach of the squangles-problem made use of a geometric representation of the triangular number of 8. When it was seen that the figure contained a part of a representation of the square number of 6 as a part, the problem was simplified to the problem of finding triangular numbers with the property that their doubles are also triangular numbers (*He has two pictures at hand*).





The intuitive inference is here again a mereological inference of the form that *a given configuration contains a part of an imaginary configuration as a part.*

LOG. I notice two things:

> (1) the part of the given whole does not need to be also given in the sense that it is explicitly delineated as a part,

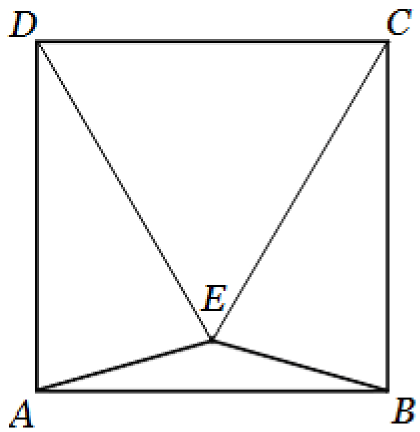
and

> (2) the whole of which the selected configuration may also be a part of is only imagined or thought.

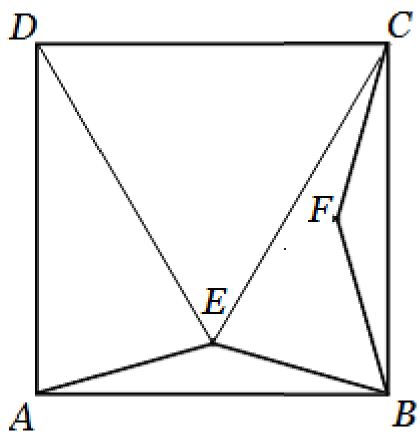
COMP. From the standpoint of a computer programmer this is remarkable. For how do we write programs in which selections take place?

MATH. This question is crucial in an example of a mereological inference which has already been presented in the story of Aristarchus' visit to Euclid. In fact it stems from my youth, at a moment when I was looking for an alternative to the 'standard' solution of the following famous problem (*He uses the blackboard over and again*):

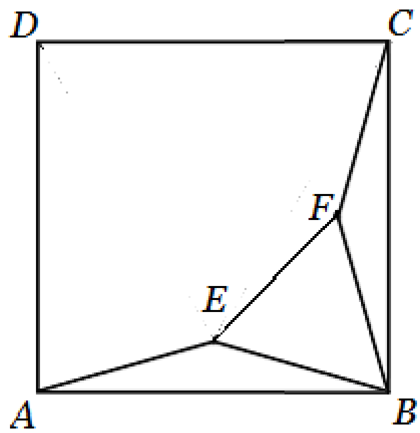
> Let the square *ABCD* be given. Within *ABCD* the point *E* has been construed such that the angles *EAB* and *EBA* are 15°. Prove that the triangle *CDE* equilateral.



I knew the 'standard' solution in which the triangle *ABE* is mirrored in the line *BD* into the triangle *BCF*, resulting in an isosceles triangle *EBF* of 60°:

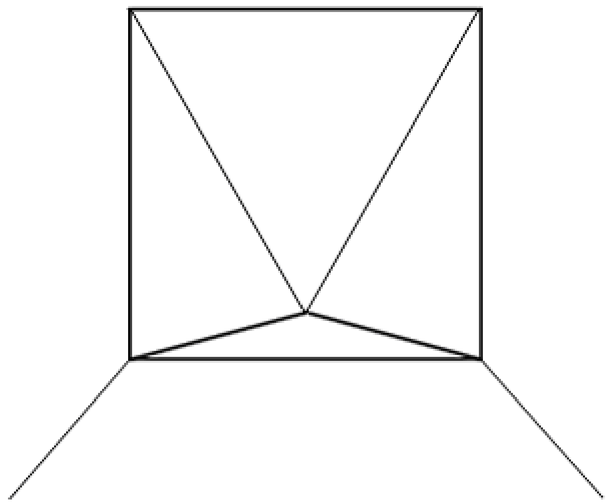


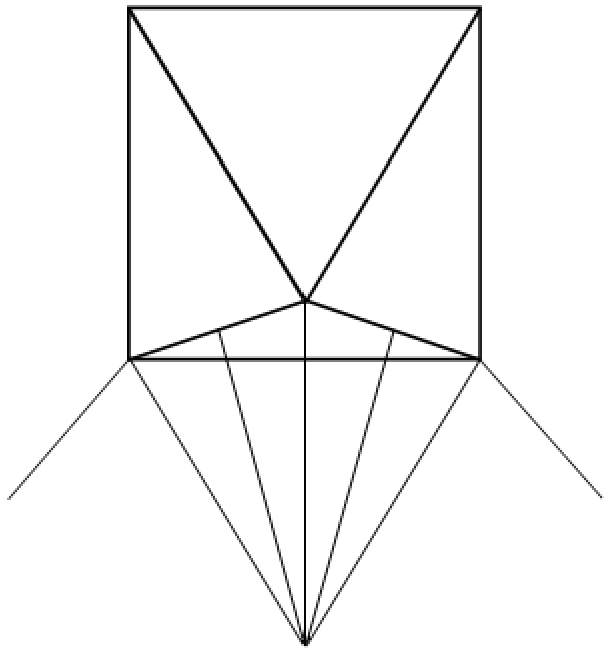
An analysis of this solution seems to yield that the mirroring was deliberately done in order to get an angle *EBF* of 60°. However it is not excluded that the symmetrical figure formed by the two congruent triangles was seen in a 'flash of insight'. In that case the line segment *BE* would have been regarded as a part of the imagined isosceles triangle *EBF* and with hindsight we would have another mereological inference:



As a schoolboy I was not acquainted with this solution, so I concentrated on the triangle *ABE*. What is so special about this triangle?

How it occurred to me I don't know, but at a certain moment I realized that the triangle *ABE* could be regarded as a chord triangle of a regular dodecagon with a circumscribed circle, so that it would make sense to construe the center of that circle. After this was done, it was not difficult any more to prove the theorem.





We see that both a mereological inference and a promising conclusion are present in this example. At the same time it becomes clear that I could only draw the intuitive inference because I was acquainted with regular polygons. In my youth these were still part of the grammar-school curriculum...

LOG. I wonder whether the significance of this example is broader than the example itself. Do mereological inferences also occur in standard approaches to traditional geometry? In other words, can some of Euclid's proofs be analyzed in such a way that they are seen to contain such inferences?

MATH. In order to answer this question, we must not look at the finished pictures of such proofs, but allow them to come about point by point, line by line, and circle by circle. I repeat: is the last example just accidental, or do mereological inferences play a substantial role in geometry? Our first extremely simple conclusion about Proposition 1 of Book 1 of Euclid's *Elements* is that here already two mereological inferences appear.

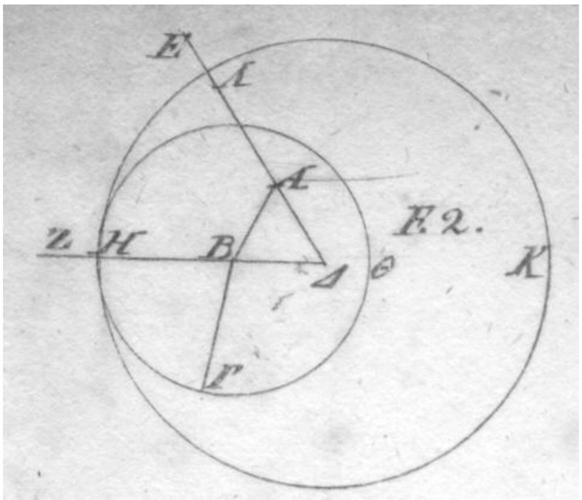
COMP. I am sorry, but don't blame me that I do not know that theorem. My knowledge of elementary geometry is almost nil, since it was abolished in secondary schools.

MATH. The proposition establishes the construction of an equilateral triangle on a given line segment. Therefore each of the end points of the line segment upon which the equilateral triangle is to be construed, must be regarded as the center of a circle. Can you imagine the two circles? Neither of the two circles is present in the picture from which the proof starts.

COMP. No problem.

MATH. It follows that we make two mereological inferences.That there is nothing mysterious about them is due to Euclid's third construction postulate: 'to describe a circle with any center and distance'. This means that any point can be regarded as the center of any circle. Experienced geometry students, who regard this as obvious, do not notice that this has been explicitly made possible by a postulate.

Things are different if we look at Euclid's proof of Proposition 2, which demonstrates a construction of a line segment with the same length as that of a given line segment 'at' a given point. It is not a trivial inference that the line segment which connects two given points – in itself already the result of a mereological inference – should be regarded as a side of an equilateral triangle. Its function becomes clear only at the end of the proof. Please have a look at the picture that accompanies this demonstration in my favorite Greek-Latin edition of Euclid's 'Elements' (*Math takes the edition by Camerer, Berlin 1824 and 1825*).



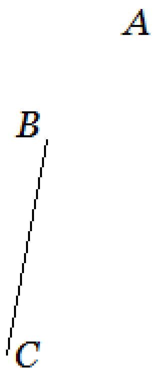
COMP. This goes too fast, can you explain how Euclid's argument goes?

LOG. Let me translate the Greek text:

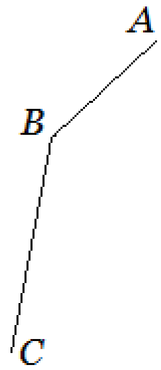
> Let the given point be *A*, and the given line segment *BΓ*. After *A* and *B* are connected, the equilateral triangle *ABΔ* is construed. Then *ΔA* and *ΔB* are prolonged in such a way that *AE* and *BZ* are formed. The circle *ΘHΓ* with center *B* and radius *BΓ* is construed. *H* is the intersection point of this circle and *BZ*. Hereafter the circle *HKΛ* with center *Δ* and radius *ΔH* is construed. *Λ* is the intersection point of this circle and *AE*. This circle intersects *AE* in *Λ*. It is easy to see that *AΛ* is equal to *BH* and hence also equal to *BΓ*.

MATH. Thank you, Log. But let me be clear about Euclid's demonstration by giving the details of the first two steps of the description, to begin with.

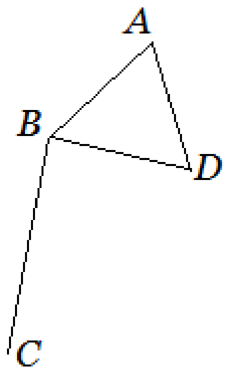
Immediately after the statement of what is given, a point *A* and a line segment *BC*,



A and *B* are regarded as the endpoints of the line segment *AB*:



Next *AB* is regarded as the basis of the equilateral triangle *ABD*:



Do you understand how the construction will be completed?

COMP. Not quite, please give a hint!

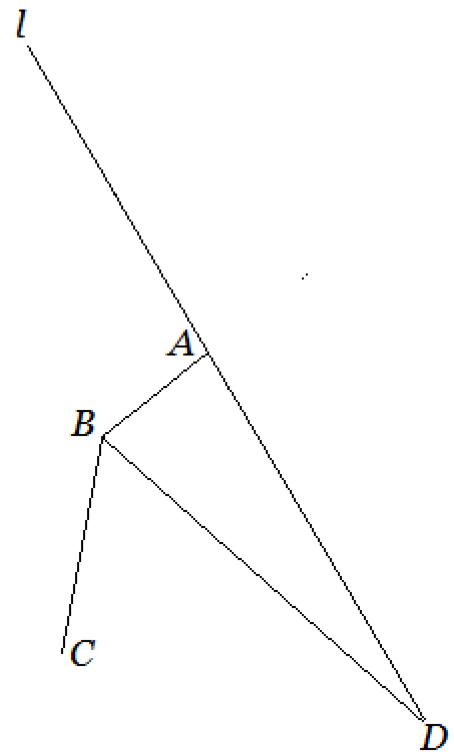
MATH. Look at the segments *DA* and *DB*. They have the same length, and we can imagine that a point *H* can be construed on the prolongation of *DB*, such that the line segment *BH* has the same length as the length of *BC*. Then we can also imagine that a point *L* can be construed on the prolongation of *DA*, such that the line segment *DL* has the same length as that of *DH*. It follows that the line segment *AL* has the same length as the line segment *BC*.

COMP. I see. And the construction makes us of a pair of compasses. That's why Euclid draw the two circles in order to pace the same lengths. This makes his figure somewhat complicated.

LOG. The theorem can be generalized to the construction of a line segment with the same length as that of a given line segment 'at' a given point and on a line throught this point, because the triangle *ABD* need not be equiliteral. An isosecles triangle with one of its sides on the given line is sufficient.

COMP. Can you draw a suitable figure?

LOG. Look:



COMP. I see, first I will onstrue *E* on the prolongation of *DB* such that *BE* is equal to *BC*, and next *F* on *l* such that *DF* is equal to *DE*. Then *AF* is equal to *BE*, and therefore also to *BC*.

MATH. Well done! This is an example of a mereological inference that *a given configuration*, here the line *l*, *contains a part of an imaginary configuration*, in this case the line segment *DE*, *as a part*.

It is not the only form mereological inferences can take. Let us look at Euclid's proof of Proposition 16,

> In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

It appears that there is the inference that a certain angle must contain another angle as a part if it is to be greater than another angle. The hypothetical part should be at least as large as the latter angle. In this case the mereological inference takes the form *that a given configuration contains an imaginary part that stands in a certain relation with another given configuration*. Similarly, in the proof of Proposition 18,

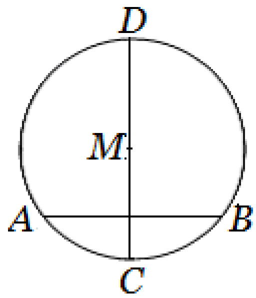
> In any triangle the angle opposite the greater side is greater,

it is assumed that a given line segment that is greater than another given line segment contains a part that is equal to the latter segment.

We can also think of mereological inferences in which only one configuration is given and hypotheses are formulated about *imaginary* parts of this configuration. Construction tasks are obvious examples. I will mention only one, Euclid's proof of Proposition 1 of Book 3,

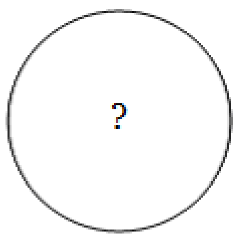
> To find the center of a given circle.

The imagined center of the circle is conceived as the midpoint *M* of an imagined diameter *CD* which in its turn is conceived as the perpendicular bisector of an imagined line segment that joins two (imagined) points *A* and *B* of the circle:



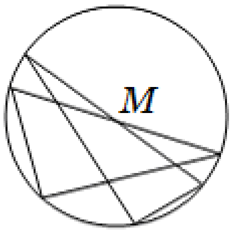
I think that most people will not find it difficult to image the total configuration without drawing a picture. Then they will readily conclude from it that the corresponding constructions are leading to the desired center. But what about the following plan:

> Think of two points on the circle and conceive each of them as the vertex of a right-angled 'chord triangle' (inscribed triangle) with the right angle at the point in question; consider the intersection of the sides opposite of the right angle; this is the desired center of the circle.



COMP. I am afraid that this is too difficult for me. Can't you give me a picture?

MATH. Here it is:



COMP. I understand that this construction must be accompanied by a proof that the intersection point is just the center of the circle?

LOG. The hypotenuses, as the sides opposite the right angle are called, are diameters of the given triangle. But as far as I know Euclid has no special theorem for this property of inscribed rectangular triangles.

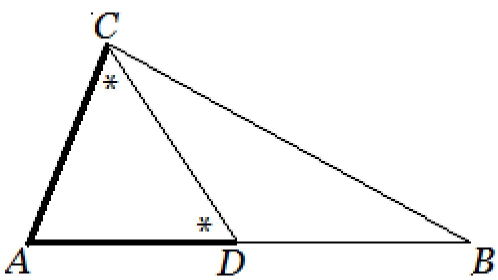
MATH. You are right, and we saw that Euclid's construction of the center of the circle is quite different from the one indicated in the last figure. Perhaps this example is not very helpful for getting a better grip on mereological inferences. The given circle has no parts at all, and the inferences seem to be guided more by known theorems about inscribed right-angled triangles, than by inspection of the circle itself.

Of course, all mereological inferences in elementary geometry depend on known theorems. Some theorems are even especially designed to support such inferences, such as Euclid's Proposition 3 which makes it possible 'to cut off from the greater of two given straight line segments a straight line segment equal to the lesser'. This theorem is used in the beginning of the proof of Proposition 18,

> In any triangle the angle opposite the greater side is greater.

Here the inference concerns, strictly speaking, an imaginary part of the greater side which then becomes a part of an imaginary isosceles triangle. There is no need to illustrate this with a figure, since it is easy to bring the situation to mind.

COMP. Nevertheless I will draw a figure:



I see, *AB* contains a line segment *AD* with the same length as AC.

MATH. This example is also revealing because of the connection with the subject of redescription. There are two descriptions involved: one that describes a line segment as part of a given configuration, and another that describes it as part of an isosceles triangle. We know that already from our discussion of redescriptions and substitutions.

LOG. Mereological inferences also occur in elementary algebra, for example when part of a formula is seen as a part of another formula:

$$x^2 + 6x = 7$$
$$x^2 + 6x + 9 = 7 + 9$$
$$(x + 3)^2 = 16$$
$$(x + 3)^2 = 4^2$$

COMP. I conclude that you give tutoring services to high school students. Can't you give a more advanced example?

LOG. I don't know if $\int(1/\sin x)dx$ counts as an example?

COMP. Obviously the question is whether the expression $1/\sin x$ can be regarded as a part of a more complex formula with which the integral can be successfully elaborated.

LOG. It can, but it looks more like a trick:

$$\int(1/\sin x)dx = \int(\sin x/\sin^2 x)dx$$

COMP. Has this been done because $\sin^2 x$ can be transformed to $1 - \cos^2 x$, an expression of the form $a^2 - b^2$?

LOG. Quite right:

$$= \int(\sin x/(1 - \cos^2 x)dx$$
$$= \int(\sin x/(1 - \cos x)(1 + \cos x)dx$$

Remember now that

$$\frac{½}{(a - b)} + \frac{½}{(a + b)} \quad \text{and} \quad \frac{a}{(a - b)(a + b)}$$

come down to the same, and apply this equivalence from the right to the left:

$$= \tfrac{1}{2}\int(\sin x/(1 - \cos x)\mathrm{d}x + \tfrac{1}{2}\int(\sin x/(1 + \cos x)\mathrm{d}x$$

Now we can integrate, because $\int(f'(x)/f(x))\mathrm{d}x = \ln f(x) + C$:

$$= \tfrac{1}{2}\ln(1 - \cos x) - \tfrac{1}{2}\ln(1 - \sin x) + C$$

When you are familiar with goniometric formulas, you can even simplify this as follows:

$$= \tfrac{1}{2}\ln(1 - \cos 2\cdot\tfrac{1}{2}x) - \tfrac{1}{2}\ln(1 - \sin 2\cdot\tfrac{1}{2}x) + C$$
$$= \tfrac{1}{2}\ln(2\sin^2\tfrac{1}{2}x) - \tfrac{1}{2}\ln(2\cos^2\tfrac{1}{2}x) + C$$
$$= \ln\tan\tfrac{1}{2}x + C$$

Conclusion: $\int(1/\sin x)\mathrm{d}x = \ln\tan\tfrac{1}{2}x + C$

COMP. Beautiful!

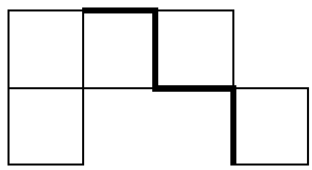
MATH. It is clear that one must know a lot, before one can find this solution. Therefore I am not sure whether the awareness of mereological inferences will as such facilitate the problem-solving process in the sense that it helps rewriting $1/\sin x$ as $(\sin x/\sin^2 x)$ right from the beginning.

LOG. Nevertheless I will recommend it in my tutoring activities!

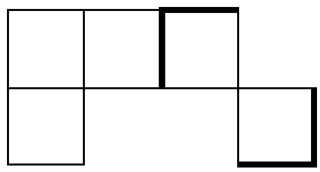
MATH. Apart from the didactical issue there is the 'philosophical' question which general abilities enable human beings to make mereological inferences. I know no better starting point in this matter than some remarks by one of my teachers, the Dutch philosopher of science E. W. Beth. After a short presentation of Kant's 'synthetic functions of the intellect', he wrote that it is very important that 'we can afterwards annihilate the synthesis formed in our mind'.[i] In contrast with the synthetic functions, this *analysis* is introspectively accessible according to Beth, who mentions two occasions on which the analytic function comes into operation. First, there is the interruption of the normal course of apperception, and second, we can bring it about voluntarily. An example of the former occurs when we listen to a melody which is suddenly broken off. In that case the analytic function breaks up the melody into a perceived and an expected part. An example of the latter occurs when hearing a melody we deliberately pay attention to some detail. It is clear that in both cases we are concerned with a simple mereological inference, and we might distinguish terminologically between a simple 'part-whole inference' and a simple 'whole-part inference', accordingly as the inference is towards a whole or towards a part.

LOG. It seems that Beth made an important theoretical discovery! It would be pleasant if we could regard every complex mereological inference as a combination of one or more of such simple inferences.

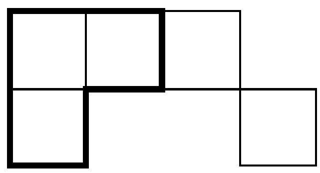
MATH. This question deserves closer inspection. In cases like our matchstick problem it is the selection of a special configuration 'within' the given configuration that matters. Of course it is possible to select an 'arbitrary' part of the given configuration:



but this may not bear upon the problem. Apparently the 'combination' of whole-part inferences and part-whole inferences is a complicated one.
Moreover, it is not necessary that the selected part should be a 'connected' configuration:



It is also possible that the selected part is not yet a part of an intended configuration, but still shows some 'defect':



In this case, the selected part itself contains a part that is part of an intended configuration. And then we are not yet talking about imagined parts that have still to be generated, as we have seen in geometrical examples.

LOG. Apparently a ‘theory’ about mereological inferences in mathematical problem solving is still far away. We must be careful not to generalize too hastily.

MATH. I agree. The example of Husserl’s mereological theory is instructive. Husserl implicitly assumed the transitivity of the part-whole relation, despite Bolzano’s warning that this relation is only transitive in a limited domain.[ii] As yet it seems wise to select a certain type of problem and try to investigate which mereological inferences are made in the solutions. For it seems undeniable that they form a separate category along with deductive, inductive, abductive, and analogical inferences.

COMP. I would like to learn more about the different kinds of inferences. Maybe next time?

MATH. On the condition that Log is also present.

LOG. With pleasure.

(*They part after they have made an appointment for their next meeting*.)

[i] Beth (1948), *Wijsbegeerte der wiskunde*. Tweede geheel opnieuw bewerkte uitgave. Antwerpen: Standaardboekhandel, p. 313-314. (*Philosophy of Mathematics*, second edition, in Dutch)

[ii] Was man zu sagen pflegt, daß die Theile eines Theiles auch Theile des Ganzen wären, gilt nur bei Inbegriffen einer gewissen Art ... Bolzano (1837), *Wissenschaftlehre*. Sulzbach: J.E. v. Seidelschen Buchhandlung, § 85.