Mendelssohn's Euclidean Treatise on Equal Temperament

Henk Visser

Moses Mendelssohn, once a member of a 'Berlin Circle', together with distinguished mathematicians such as Euler, was, like him, interested in 'mathematical music'.[1] But different from Euler, who was an expert in 'natural' tonal relationships, Mendelssohn wrote about an artificial way of solving the problem of intonation for organs, harpsichords and clavichords. He connected solutions of the ancient Greek Delian problem (of constructing distances with the length of the cubic root of a given length) with the problem of equal temperament tuning. Thereby he gave an original proof for Newton's solution.

Mendelssohn's method was not adopted in practice, for two reasons: the reluctance of musicians to tune their instruments with an equal division of the octave at all, and the alleged practical difficulty of working with a monochord on the basis of Mendelssohn's approximations.[2] As a result, only the title of Mendelssohn's article was mentioned in the musico-theoretical literature, without a closer inspection of its content. And his mathematical contribution was ignored by mathematicians who have extensively written on the subject.[3] (Conti 1900 and 1907).

For a proper appreciation of Mendelssohn's mathematical talents, an analysis is given of the way he proved Newton's theorem with Euclidean geometrical means. This requires some geometrical knowledge and familiarity with the algebraic theory of proportions, but I will try to explain Mendelssohn's proof in a hopefully lucid way.

I will end with a short discussion of the question whether equal temperament tuning is compatible with Mendelssohn's aesthetics of music. The lecture begins with a short introduction, in which the need of special tuning ways is expounded.

Introduction

In the course of the eighteenth century, the development of classical music reached a point at which performances on instruments with fixed tones became more and more intolerable. Transitions from one key to another required changes in pitch which organs and harpsichords could not follow. Cavallo confirmed this when he wrote :

> When the compositions of old masters are performed in concert, and with the organ or harpsichord tuned in the common manner, the effect is

[1] Kayserling, *Moses Mendelssohn*, p. 75.
[2] Fischer, *Geschichte der Physik*, p. 254.
[3] Conti, *Problemi di 3.^{0} grado* and Conti, *Aufgaben dritten Grades.*

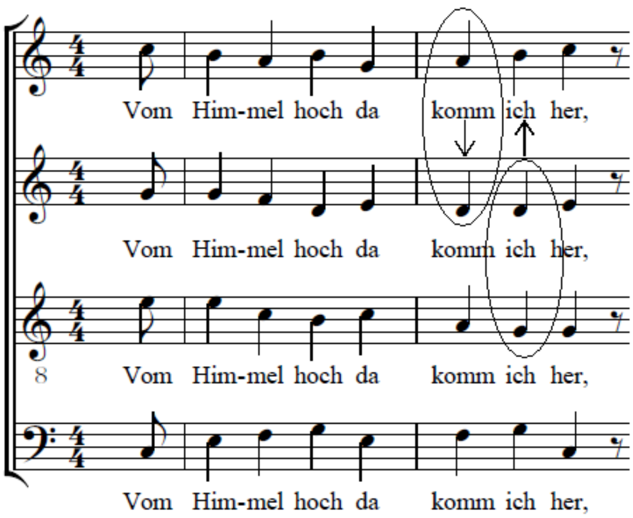
> frequently disagreeable. This is particularly the case with the songs of HANDEL, GALLUPPI, LEO, PERGOLESE, and others, who wrote in a great variety of keys, and very often in those, for which the common way of tuning is not at all calculated.[4]

Musical persons can take subtle intonation differences into account when they sing a capella. This holds as well for unison singing as for part-singing. For example, when you and I start singing the following tune – see Example 1[5] – we will intonate the first *e''* slightly lower than the second:

Example 1

A similar effect can be found in the well-known choral melody 'Herzliebster Jesu'– also known as 'Father, most holy and Suffering Savior' – by Crüger (after Schein), which J. S. Bach used twice in a modified form in the St. Matthew Passion and in the St. John's Passion. How this has come about, is a question that we can leave aside.

Suppose further that we have a professional choir at our disposal, and ask the singers to perform the following four-part version of the beginning of another well-known choral, see Example 2:[6]



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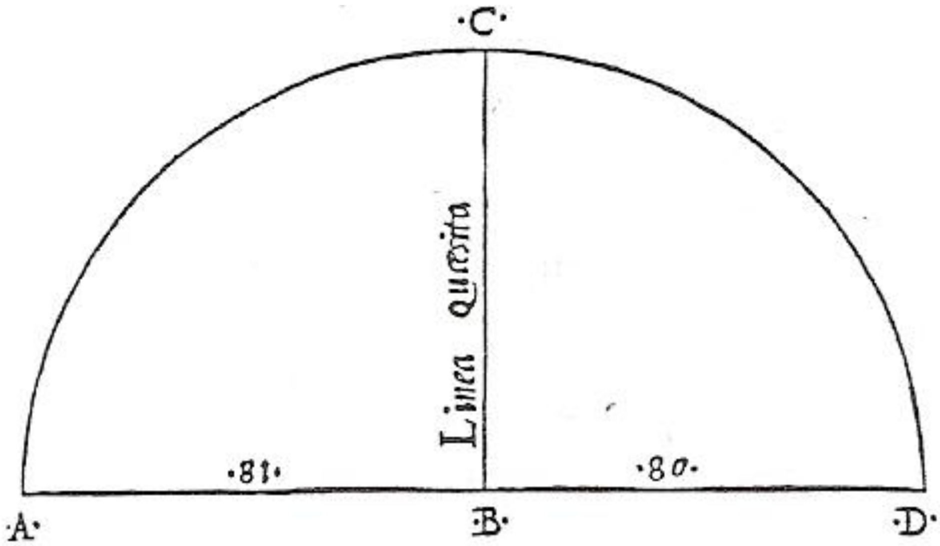
[4] Cavallo, *Of the temperament*, 253-254.
[5] Valerius, *Nederlandtsche Gedenckklank*, p. 168. The English text is as follows: 'We gather together to ask the Lord's blessing; He chastens and hastens His will to make known.'.
[6] Harmonization attributed to Hassler.

Example 2

It appears that the altos can intonate the second *d'* in the second bar slightly higher than the preceding *d'* if they feel that the first *d'* must form a pure fifth with the *a'* of the sopranos, and the second *d'* a pure fifth with the *g* of the tenors. In general, experienced singers 'can make small tuning adjustments quickly'.[7]

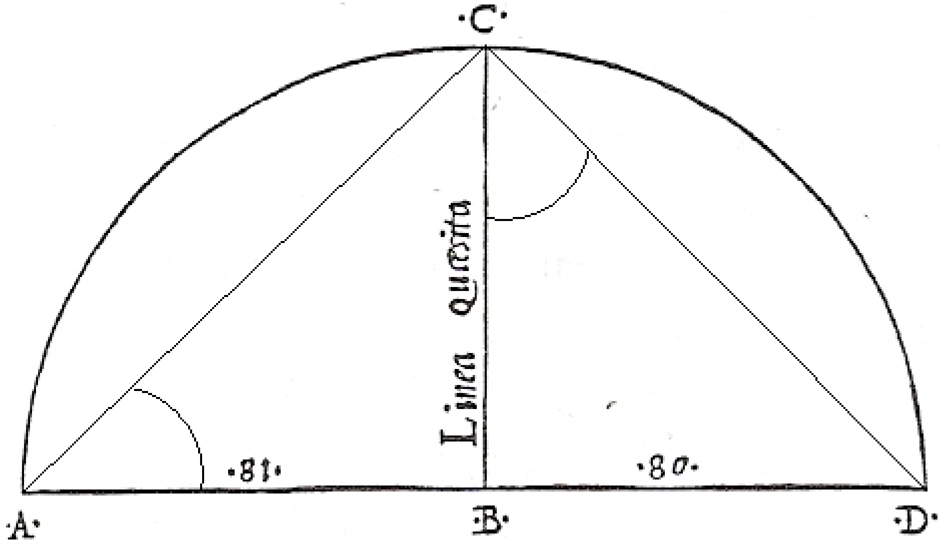
However, when the same musical fragments are played on an organ, harpsichord or clavichord, these different intonations are not possible, given the obvious limitations of these instruments. They have as a rule only one pipe or string available for two different tones with the same name. What must be done when there is only one string available for the two different *d"*s of Example 2? In 1529, the Italian choirmaster Ludovico Fogliano proposed to tune the string with the geometrical mean of the two *d"*s. The length of the corresponding string can be construed with a geometrical method . Given that the frequency ratio of the lower *d* and the higher *d* is 80 : 81, the task is to find the number *x* such that 80 : *x* = *x* : 81. It is Euclid who indicated how its magnitude can be geometrically determined. This is shown in Example 3.[8]



Example 3

It follows from the similarity of the triangles *ABC* and *CBD* that *AB* : *BC* = *CB* : *BD.* See Example 4.

[7] Cathérine Schmidt-Jones on internet.
[8] Lindley, *Stimmung und Temperatur*, p. 142.



Example 4

Better known are non-mathematical solutions such as the meantone temperament. In this tuning system, preference is given to pure major thirds at the cost of fifths. Thereby such musical pieces as the preceding examples sound not too bad, although the interval *d' a'* of Example 2 is smaller than a pure fifth, and the interval *g d'* larger than a pure fourth. It follows that composers took into account the sounding results when they wrote explicitly for keyboard instruments with mean tone tuning. However, as soon as there are too many sharps, the results are unbearable.

For this reason, compositions written by composers who wanted to switch from one key to another, no problem for singers, became more and more difficult to perform on keyboard instruments. Therefore musicians tried to find methods in order to solve this problem in the most satisfactorily way, not only with regard to the euphony of the results, but also to the feasibility of the tuning in practice.

Or is there a mathematical way of theoretically solving all problems simultaneously, by dividing the octave into twelve equal parts, that is to find eleven mean proportionals ('*middelevenredigen* ') between 1 and 2? This generalization of Fogliano's approach was described by Simon Stevin circa 1595, although Stevin was not aware of the subtle differences for which Fogliano invented his solution.[9] Stevin derived the lengths of the corresponding strings arithmetically. It is clear that the outcomes are only approximations.

The importance of the calculations seems to be that frets can be added to a monochord in the appropriate places in order to make the different pitches audible. The discussion of this practical application will be postponed until after a discussion of Moses Mendelssohn's approach.

[9] Dijksterhuis, *Simon Stevin*, p. 270-276.

Mendelssohn's contribution

It is well-known that Mendelssohn was a serious student of mathematics until 1760.[10] Euclid's *Elements* and Newton's *Principia* are two of the works he thoroughly explored. It was one of the ways he compensated for his lack of a gymnasium education. But different from most abiturients he did not leave it at that. His participation in the *Berliner Kreis* ('Berlin Circle') of mathematicians and other scientists may have stimulated him to present more mathematical work than his treatise on probability. It is also possible that discussions on musico-theoretical subjects instigated by another famous member of the Circle, Euler, gave Mendelssohn the idea of combining the ancient Delian problem of doubling the cube with the question of equal temperament. He worked it out in his 'Versuch, eine vollkommen gleichschwebende Temperatur durch die Construction zu finden'.[11]

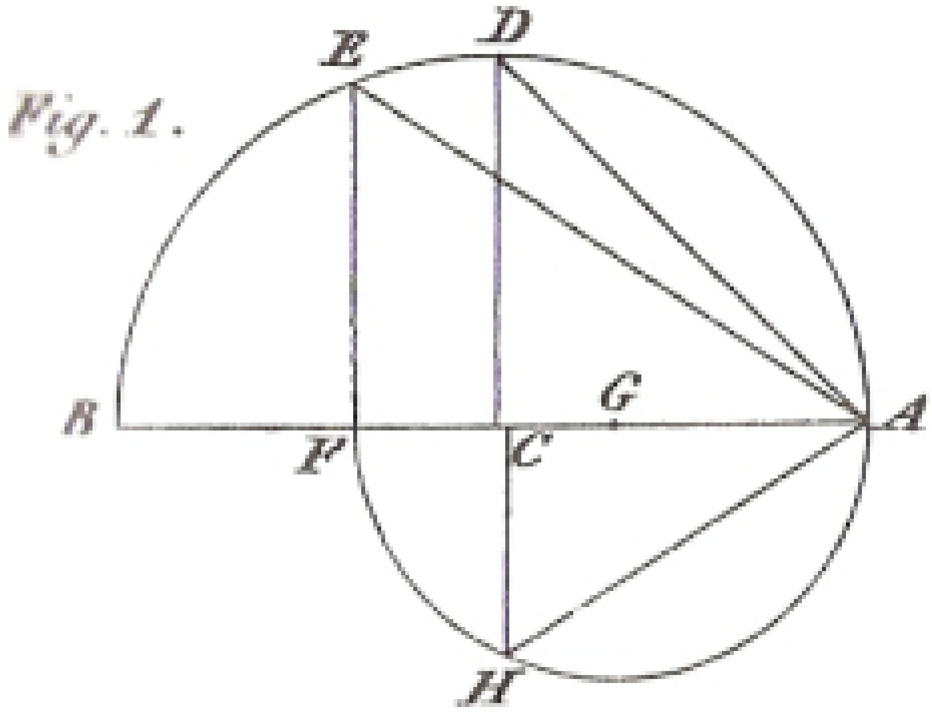
Mendelssohn noticed that the problem of the division of the octave into twelve parts by eleven mean proportionals can be split into three subproblems:

> (1) the division of the octave into two parts by one proportional,
>
> (2) the division of each of these two parts again into two parts by one proportional,
>
> (3) the division of each of the resulting four parts into three parts by two proportionals.

The third subproblem is equivalent to the Delian problem: if $a : x = x : y$ and $x : y = y : b$, then $x^3 = a^2b$ and $y^3 = ab^2$.

The first two subproblems present no difficulties: a geometrical solution has already been given in Example 4, Fogliano's method. Mendelssohn brought his solution of the two problems into one figure. See Example 5.[12] I will give it presently, without much commentary, because Mendelssohn's approach is in principle the same as Fogliano's.

[10] Kayserling, *Moses Mendelssohn*, p. 75.
[11] *Marpurgs historisch kritische Beiträge*. Band 5, St. 2, 1761, p. 95-109; Mendelssohn , *Gesammelte Schriften Band 2*, p. 189-199.
[12] Mendelssohn, *Gesammelte Schriften Band 2*, p. 197, corrected.



Example 5

Mendelssohn postulated that the interval *c c'* should be divided into parts in the following way:

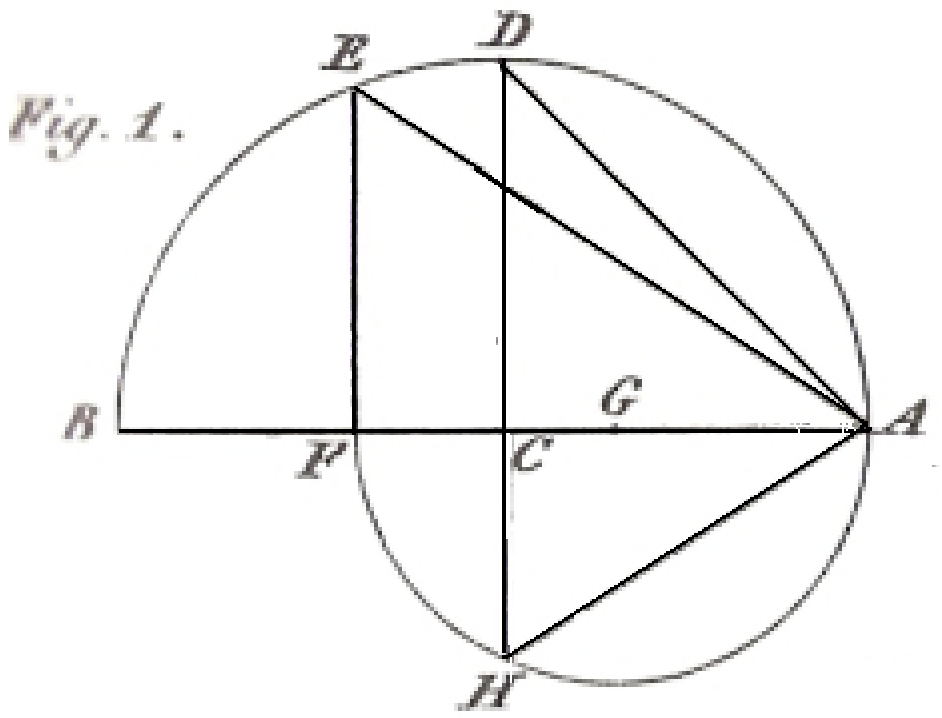
c cis d dis e f fis g gis a bes h c'

With an equal temperament the frequency of the tone *fis* is exactly the geometric mean between the frequencies of *c* and *c'*:

c *cis d dis e f* ***fis*** *g gis a bes h* ***c'***

The same holds for the length of its string and the lengths of the strings of c and *c'*.

This leads to Mendelssohn's solution of the first two subproblems. Unfortunately, his Figure 1 is misleading, since it looks as if there are two points *C*. This has been corrected in Example 6.



Example 6

Suppose that the length of a string with height pitch *c* is *AB*, then the length of *c'*, an octave higher, is half of it, *AC* in Mendelssohn's Fig. 1.

Let *CD* be the perpendicular of *AB*, with *C* the midpoint of *AB*, and *D* on the circle described with *AB* as its diameter. Then *AB* : *AD* = *AD* : *AC*, and *AD* is the length of the string of *fis*.

The frequency of the tone *dis* is exactly the geometric mean between the frequencies of *c* and *fis*:

c *cis d* ***dis*** *e f* ***fis*** *g gis a bes h c*

If we construe *F* on *AB*, such that *AF* = *AD*, and make *FE* the perpendicular of *AB*, *E* on the circle, then *AB* : *AE* = *AE* : *AF*, and *AE* is the length of the string of *dis*. Similarly *AH* becomes the length of the string of *a*, because *AF* : *AH* = *AH* : *AC*.

c cis d dis e f ***fis*** *g gis* ***a*** *bes h* ***c'***

It remains to show that *AE* : *AF* = *AF* : *AH*. This can easily be forgotten, but Mendelssohn saw it, and it shows that he was a good mathematician. His proof is ingenious, because it is surprisingly done by reformulating the equations

$AB : AE = AE : AF$ and $AF : AH = AH : AC$ respectively as $AB : AF = AE^2 : AF^2$ and $AF : AC = AH^2 : AC^2$. The desired conclusion follows with help of the equation $AB : AF = AF : AC$.[13]

Of course, the third subproblem is the crucial step. It asks for two mean proportionals, for example between *c* and the new obtained *dis*:

c *cis d* ***dis***

Mendelssohn's view of this problem is remarkable: he regarded it as a relatively simple task, given that so many ancient authors had solved it:

> Therefore the real point is the well-known *Delian problem*, which has caused so much stir in Antiquity.[14]

He mentions eight Greek scientists, and refers to Eutocius' Commentary on Archimedes for their solutions.[15] So why was it necessary to go deeper into the matter? Mendelssohn gives three arguments:

> (1) some of the Ancient solutions are based on curves, and therefore not useful for his purpose: to give a practical method for construing the different string lengths of equal temperament;
>
> (2) there is a better method than the Ancient solutions that use only ruler and compasses, namely Newton's construction;
>
> (3) Newton did not prove the correctness of his solution.

Obviously, the third argument formed a challenge for Mendelssohn. Presumably he was the first to do what Newton omitted. But in order to show the contrast with Newton's method, he presented first the Ancient solution attributed to Heron. See Example 7, Mendelssohn's Figure 2.

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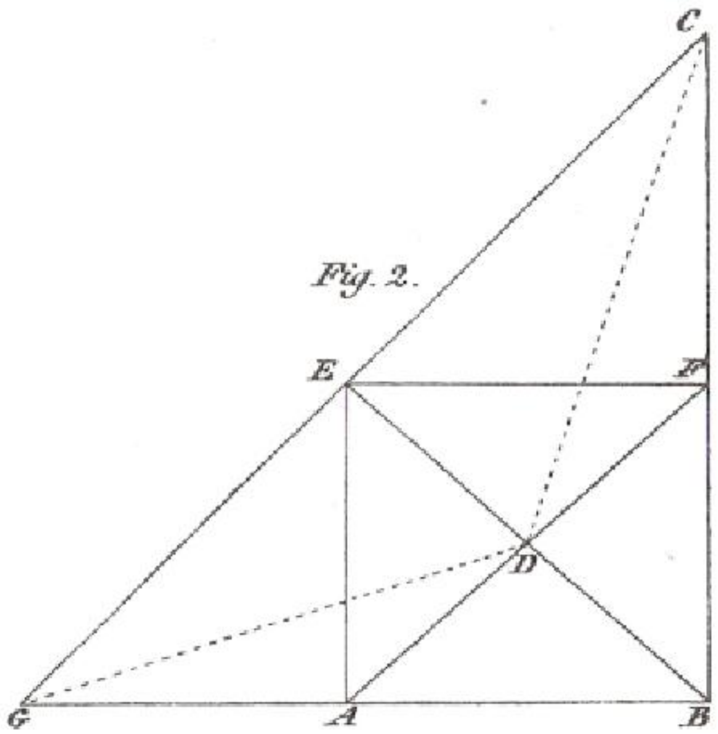
[13] Perhaps it is easier to write AE as $\sqrt{AB \cdot AF}$ and AH as $\sqrt{AF \cdot AC}$. Then $AE \cdot AH = AF\sqrt{AB \cdot AC}$. From $AB \cdot AC = AF^2$ it follows that $AE \cdot AH = AF^2$, and therefore $AE : AF = AF : AH$.

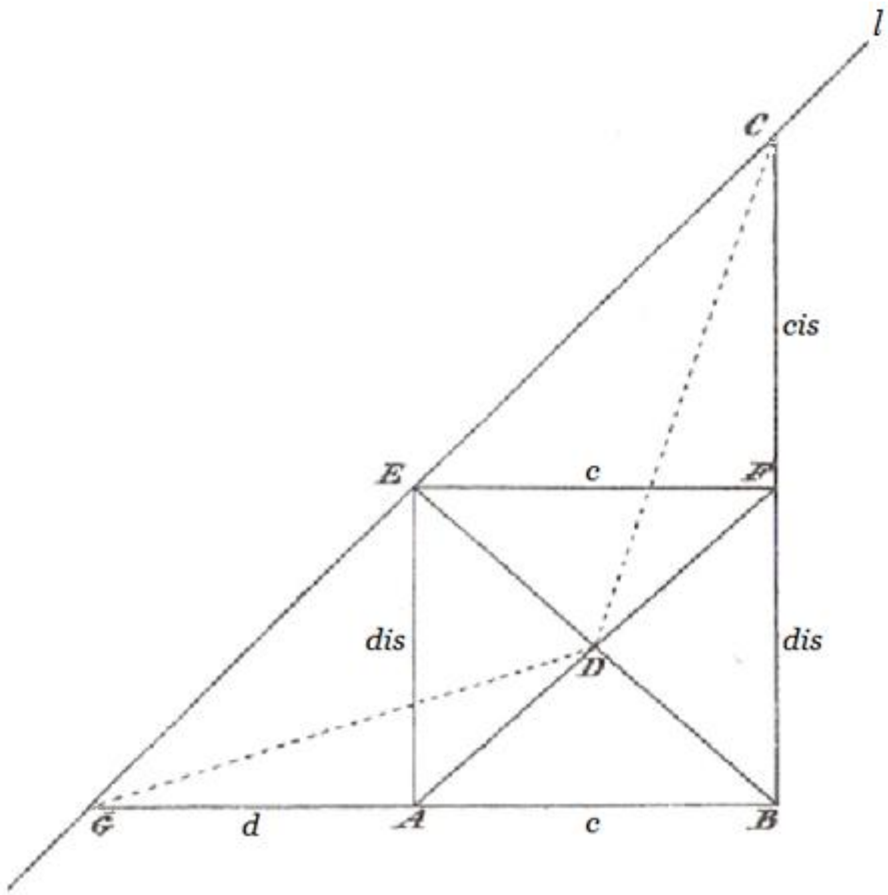
[14] Mendelssohn, *Gesammelte Schriften Band 2*, p. 192: Es kömmt also blos auf das bekannte *problema deliacum* an, das in dem Alterthum so viel Aufsehens gemacht hat.

[15] Reproduced in Thomas (ed.), *Selections*, Chapter IX.1. Duplication of the Cube.



Example 7

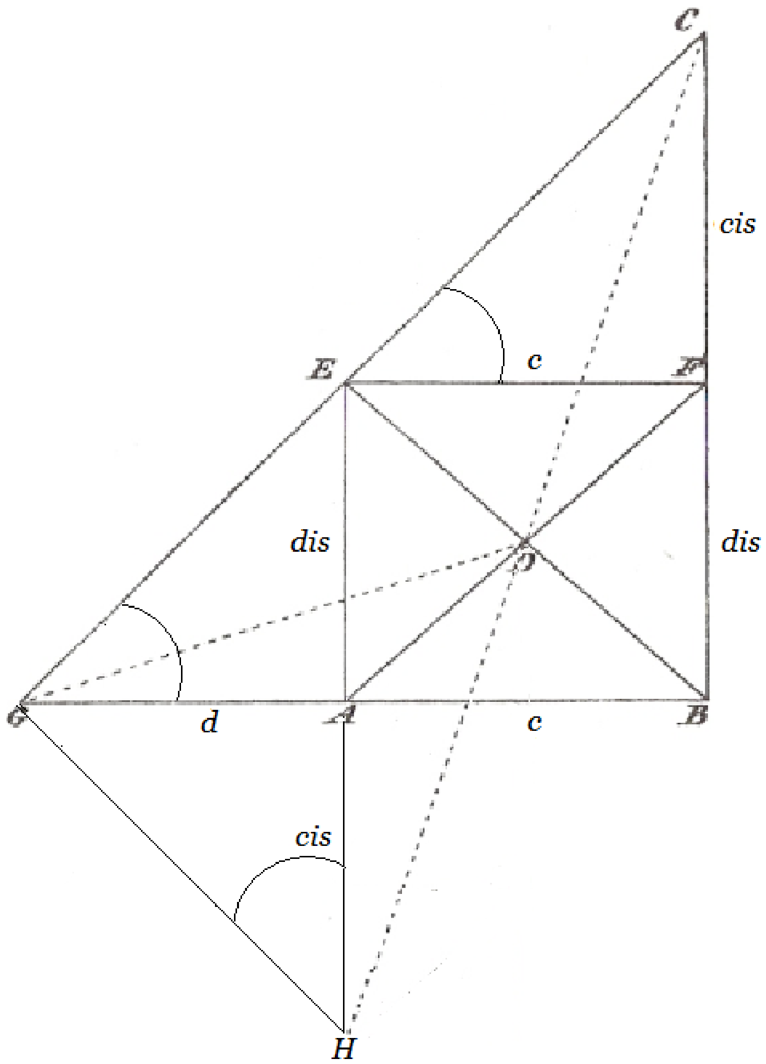
Mendelssohn’s figure, slightly extended, shows the construction. See Example 8.



Example 8

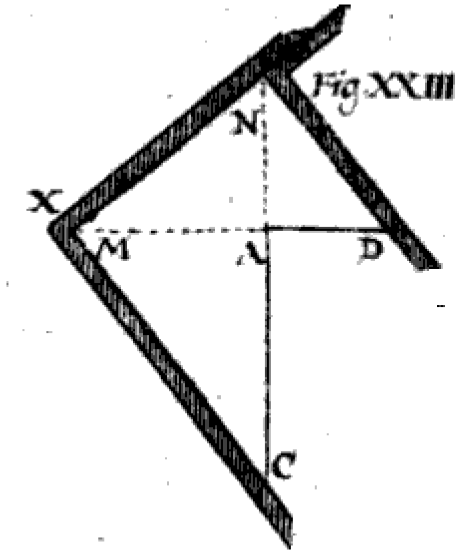
Let *AB* be the length of *c*, and *AE* that of *dis*. *D* is the intersection of the diagonals of the rectangle *ABFE*. A line *l* through *E* intersects the prolongation of *BF* in *C*, and the prolongation of *BA* in *D*, such that *DC* = *DG*. (In practice, this line must be determined experimentally, because the equality of *DC* and *DG* cannot be achieved by a construction.) The claim is that the length of *CF* is equal to *cis*, and the length of *GA* equal to *d*, and that *c* : *cis* = *cis* : *d* and *cis* : *d* = *d* : *dis*.

One sees immediately that *c* : *cis* = *d* : *dis*, but clearly this is not enough. Fortunately it easily proved that *c* : *cis* = *cis* : *d*, or that *cis* : *d* = *d* : *dis* by prolonging *CD* and *EA* to their intersection *H*. Then *HD* = *CD*, and the triangle *CGH* is rectangular in *G*. Then there are three similar triangles. They give the desired proportions. See Example 9.



Example 9

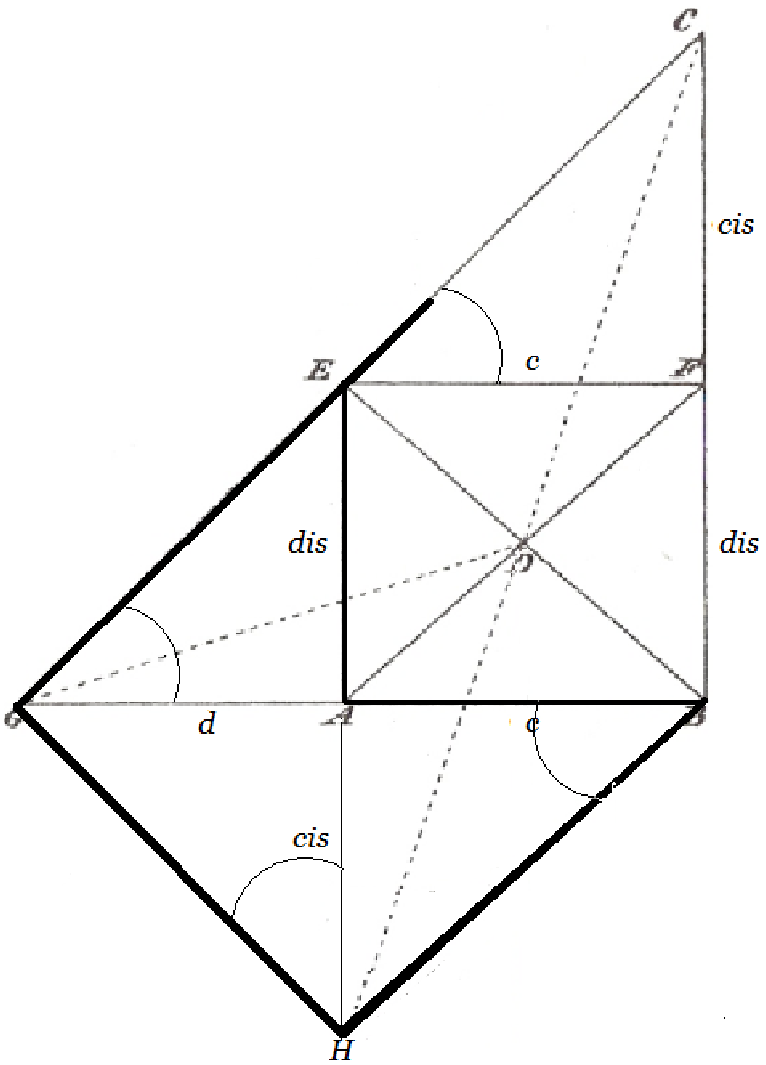
Mendelssohn referred for a proof to Sturm. It seems that he meant Sturm’s account in his edition of the works of Archimedes. But here is Sturm’s method, as it is demonstrated in his *Mathesis Iuvenilis*.[16] He uses a carpenter’s square with one movable leg. See Example 10.



Example 10

Sturm’s figure can easily be fitted into the figure of Example 9. The trick is to get a rectangle *EGHB* such that *G* lies on the prolongation of *BA* and *H* on the prolongation of *EA*. See Example 11.

[16] Sturm, *Mathesis iuvenilis*.



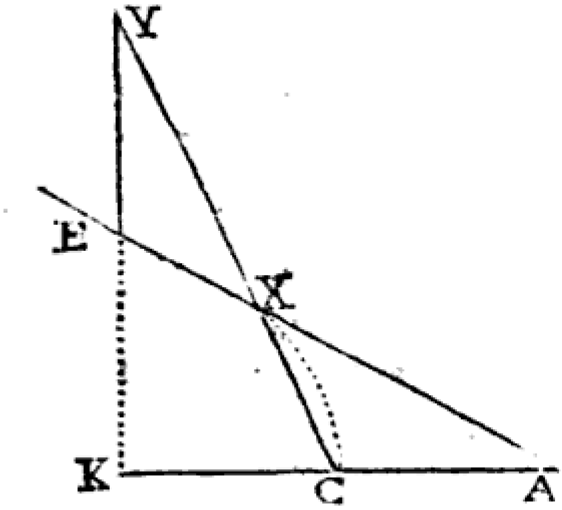
Example 11

Mendelssohn could have known this procedure from Eutocius' Commentary. Nevertheless he mentioned only the way to find the place where the place is to be found where *DC* = *DG*. He called this way 'mechanical', because the ruler cannot be surely (*sicher*) placed.[17] But of course, Sturm's procedure, attributed to Plato, is also mechanical. The same holds for Mendelssohn's prescription, which he may have preferred because it only requires a ruler with a scale indication. He based it upon a fragment he found in Newton's *Arithmetica universalis*.[18]

The problem Newton dealt with was to find two mean proportionals x and y between two given distances a and b in the sense that $a^2 : x^2 = x : b$ and, apparently also $a : y = y^2 : b^2$, the Delian problem. Then we have not only $a : x = y : b$ but also $a : x = x : y$ and $x : y = y : b$ ('Invenienda sit inter *a* & *b* duae media proportionales *x* & *y*. Quoniam sunt *a, x, y, b* continue proportionales erit

[17] Mendelssohn, *Gesammelte Schriften Band 2*, p. 193.
[18] Newton, *Arithmetica Universalis*, First edition, p. 303-304.

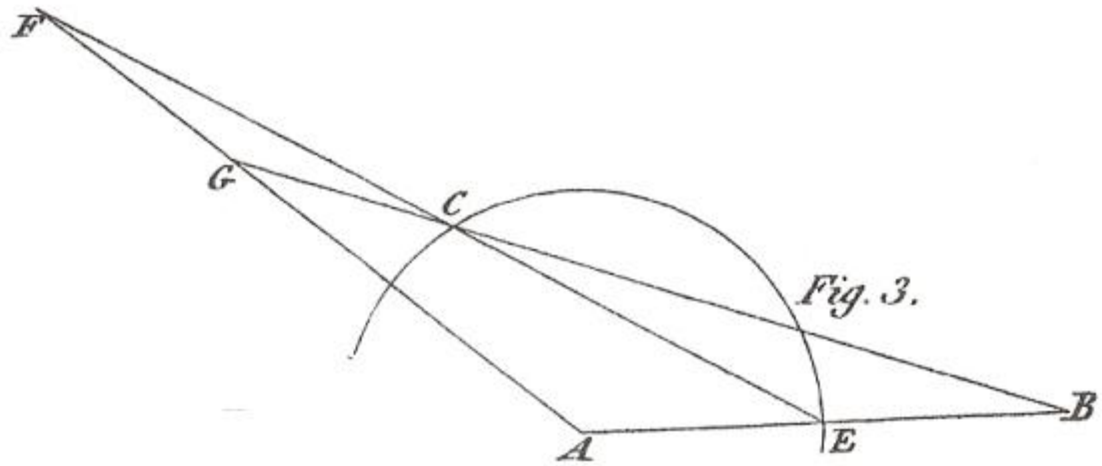
a^2 ad x^2 ut x ad b, adeoque $x^3 = aab$, seu $x^3 - a^2b = 0$’). His figure is reproduced as Example 12.



Example 12

It is supposed that $KA = a$ and C divides KA in two equal parts. X lies on the circle with midpoint K and radius KC such that $CK = b$. The lines AX and CX are infinitly produced (‘infinite productas’), but a line from K is drawn such that it cuts a line segment $EY = CA$ (=½a) from these lines. Then it follows that KA, XY, KE, and CX form a progression in the sense that XY and KE are the two mean proportionals between a and b (‘duae mediae proportionales inter a & b’). The construction is known, least to Newton, who wrote: ‘Constructio nota est’.[19]

Mendelssohn quoted the last sentence, and he gave the following perspicuous figure (Example 13):[20]

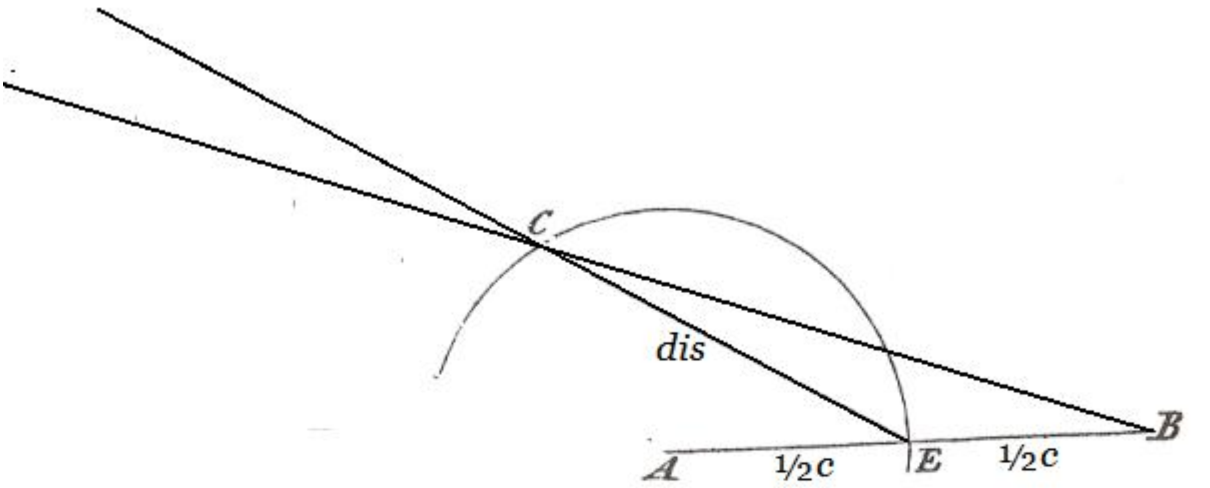


[19] Mendelssohn, *Gesammelte Schriften Band 2*, p. 193.
[20] Mendelssohn, *Gesammelte Schriften Band 2*, p. 197.

Example 13

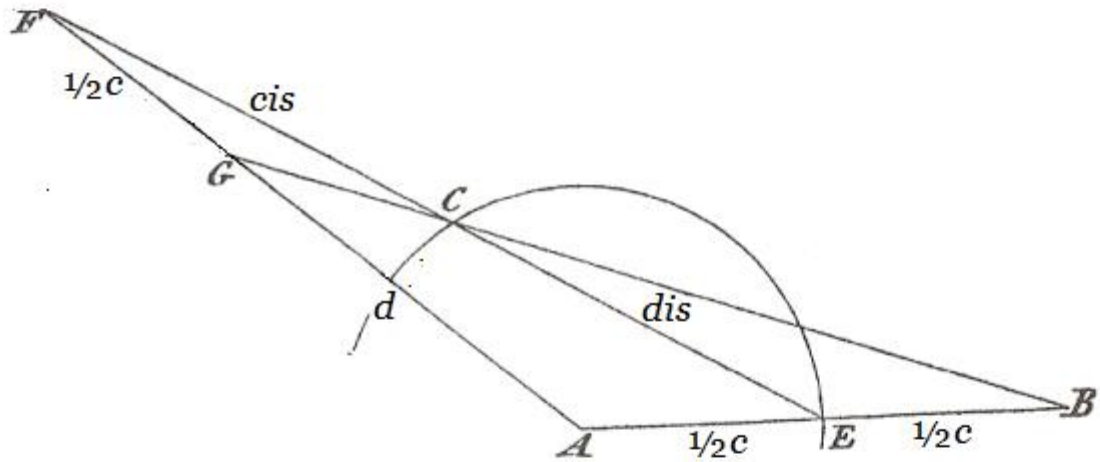
It can be described in terms of Mendelssohn's problem of finding the mean proportional of *c* and *dis*:

First, three points, *A*, *B*, *C*, and *E*, are given such that *AB* represents the length of *c*, and *EC* the length of *dis*, whereas *E* is the midpoint of *AB*. See Example 14.



Example 14

Then a line through *A* is drawn, such that it cuts a segment of length ½*c* from the prolongations of *BC* and *EC*. The claim is that *CF* becomes the length of cis, and *AG* the length of *d*. See Example 15.



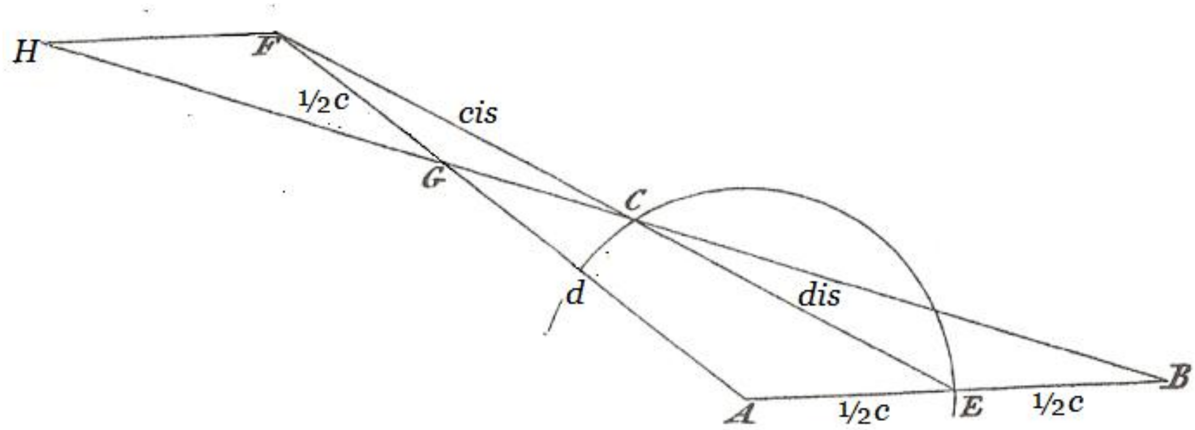
Example 15

In order to prove this, an auxiliary line must be drawn in order to get similar triangles. There are (at least) two possibilities,

(1) a line through *F* parallel to *AB*

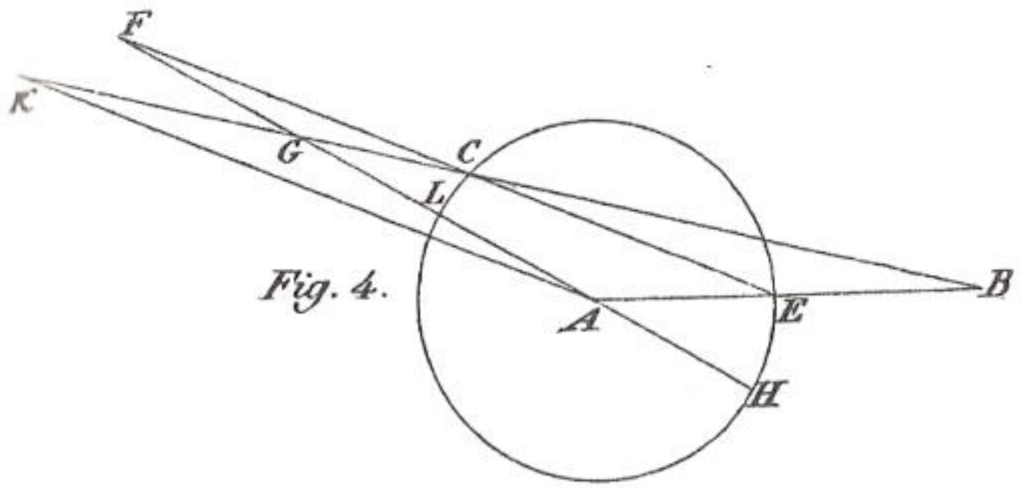
(2) a line through *A* parallel to *EC*

The line through *F* parallel to *AB* lies at hand, at least for contemporary geometricians, see Example 16.



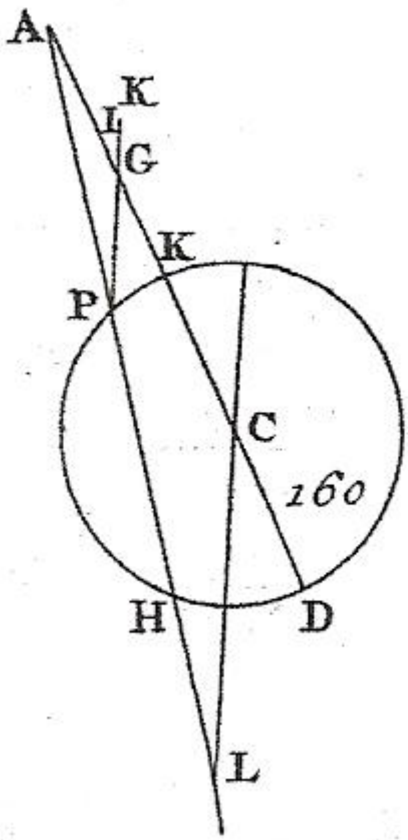
Example 16

It is interesting that Mendelssohn draws the line through *A* parallel to *EC*. See Example 17.



Example 17

It seems that there is still another possibility, invented by the Dutch geometrician van Swinden in his *Grondbeginsels der meetkunde*.[21] See Example 18.[22]

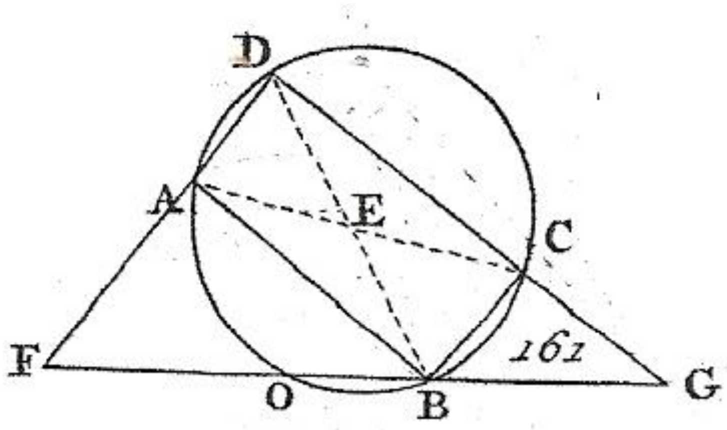


Example 18

However, van Swinden came no further than *KD* : *AP* = *AK* : *PH*, which amounts to the same as *c* : *cis* = *d* : *dis*. However, from this equation it does not follow that *c* : *cis* = *cis* : *d*, as one can see from the following simple arithmetical example: 1 : 2 = 3 : 6, but 1 : 2 ≠ 2 : 3.

Van Swinden did not manage to prove 'the missing link' *c* : *cis* = *cis* : *d*. He fell back on Heron's construction in order to prove this equation.
See Example 19.[23]

[21] Van Swinden, *Grondbeginsels der meetkunde*.
[22] Van Swinden, *Grondbeginsels der meetkunde*, Tabel IV.
[23] Idem.



Example 19

The fact that Mendelssohn did find the complete solution shows that he was a genuine mathematical talent. It is not easy to prove the complete formula, that is, in musical terms, *c* : *cis* = *cis* : *d* = *d* : *dis,* and not only *c* : *cis* = *d* : *dis,* from Newton's figure. The missing link requires the application of two extra theorems, and Newton would certainly not have attained the end 'in one step', as Mendelssohn suggested:[24]

> Allow me to prove what *Newton* presupposes as being well-known. Great geniuses attain the aim in one step, towards common minds must be guided by a long series of deductions.

There is reason to assume that Newton left the proposition without proof because it would require an undesirable extensive explanation.[25]

Back to Mendelssohn's proof. What follows is a rational reconstruction: I give the relevant line segments the names of the musical tones in order to show more clearly why the extra theorems are needed.

It is easy to see that *AK* = 2·*CE,* and hence *AK* = 2·*dis.* See Example 20.

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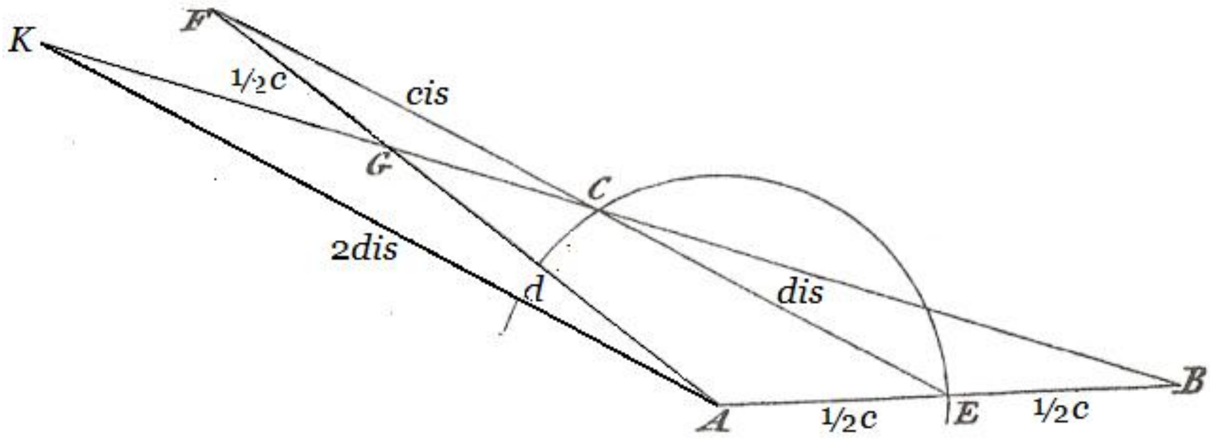
[24] Mendelssohn, *Gesammelte Schriften Band 2*, p. 193: Es sey mir erlaubt, dasjenige zu beweisen, was der *Newton* als bekannt voraus setzt. Große Genies erreichen das Ziel mit einem Schritt, wohin sich gemeine Geister durch eine lange Reihe von Schlüssen müssen leiten lassen.

[25] Newton, *Arithmetica Universalis,* First edition, p. 279: 'Demonstrationes non semper adjunxi quoniam satis faciles mihi visae sunt, & nonnunquam absque nimiis ambagibus tradi non possent. This quotation was brought to my attention by Dr. Edith Sylla.



Example 20

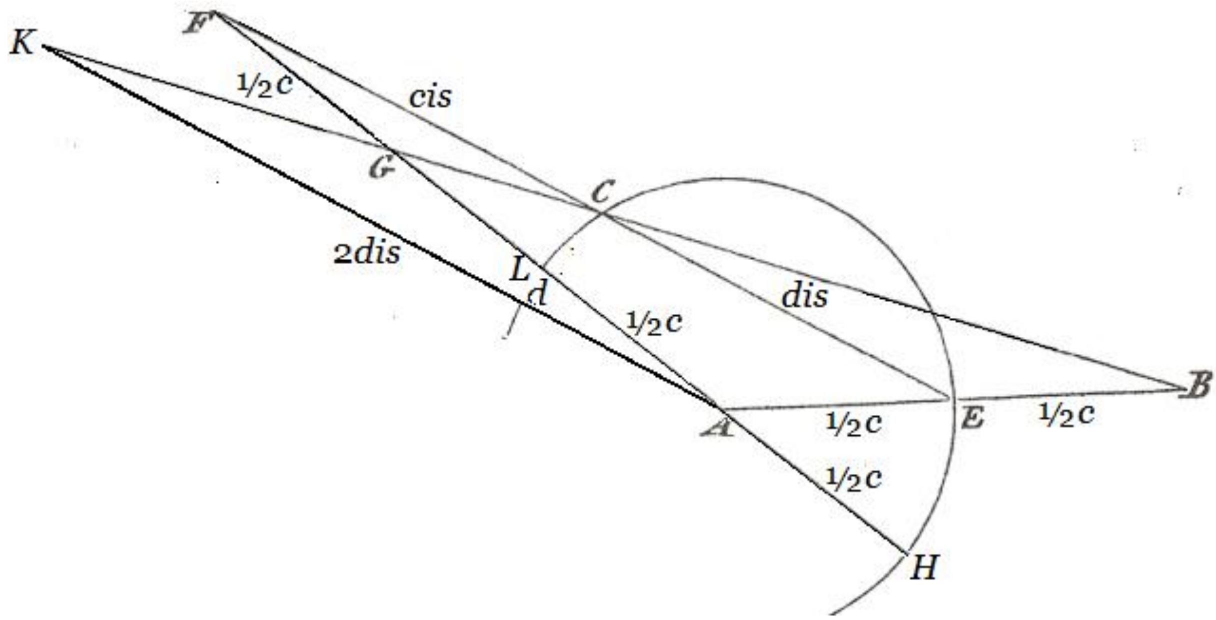
Next, it follows from the similarity of the triangles *FGC* and *AGK*, see Example 28, that *FG* : *FC* = *AG* : *AK*, hence ½*c* : *cis* = *d* : 2·*dis*, and therefore

$$c : cis = d : dis$$

So far, so good. But it is not enough, as we have seen. Perhaps the application of a well-known theorem is helpful.

$$FC \cdot FE = FL \cdot FH$$

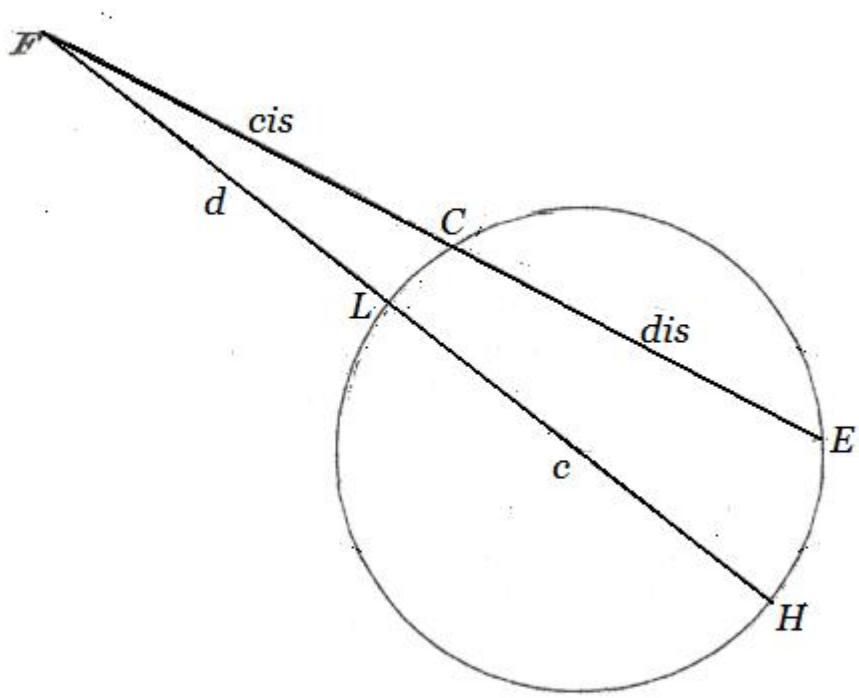
See Example 21.



Example 21

It is a corollary of Proposition XXXVI of Book III of Euclid's *Elements*, nowadays known under the name 'Power of a Point Theorem'. Mendelssohn mentions Proposition XXXVII, but that is incorrect.

See Example 22 for a simple figure for the conclusion that $FC \cdot FE = FL \cdot FH$.



Example 22

That the length of *FL* is *d*, is easy to see from Example 21. Therefore we have

$$cis(cis + dis) = d(d + c)$$

or, if you wish,

$$cis(cis + dis) = d(c + d)$$

Because an equation of the form $p \cdot q = r \cdot s$ is equivalent with $s{:}q = p{:}r$ for nonzero values, this leads to

$$(c + d) : (cis + dis) = cis : d$$

What worries us are the sums (*cis* + *dis*) and (*c* + *d*). Fortunately, a theorem about proportions provides us with these sums, because it allows us to draw a suitable conclusion from

$$c : cis = d : dis,$$

namely

$$(c + d) : (cis + dis) = c : cis$$

This is again based on a theorem of Euclid's Elements, namely Proposition XII of Book V. It can easily be proved by an application of the above mentioned equivalence, but a numeric example may be helpful:

$$1 : 2 = 3 : 6 \text{ leads to } (1 + 3) : (2 + 6) = 1 : 2$$

Combining the two new results gives

$$c : cis = cis : d$$

Now we have both

$$c : cis = cis : d \text{ and } cis : d = d : dis,$$

or, as is often written:

$$c : cis = cis : d = d : dis$$

Hereby Newton's account is completed. Curiously, as late as 1907 the Italian mathematician Alberto Conti[26] felt urged to reproduce a proof given by Carrara,[27] in which an appeal was made to Ceva's theorem, because Newton 'left the proposition without proof' (*Newton läßt diese Behauptung ohne Beweis.*[28])

With Mendelssohn's proof of the correctness of Newton's prescription, a monochord can be construed in such a way that the frets are just in the places for the tones of the equal temperament of the octave. However, it is the result of a 'mechanical' construction, and this means that one must hope that the concrete outcomes correspond with the abstract desiderata.

That it is possible to achieve good results, using a monochord with a very exact subdivision (*un monocorde muni d'un division très-exacte*[29]), seems to appear from the following report by Biot of a successful experiment:[30]

> Cavallo, exact and ingenious physicist, reports in the Philosophical Transactions, having carefully tuned an ordinary harp on these principles,

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[26] Conti, *Aufgaben dritten Grades*, an extended version of Conti, *Problemi di 3.*^{0} *grado*.
[27] *Sui tre problemi classici degli antichi in relazione di recenti risultati della scienza.*
[28] Conti, *Aufgaben dritten Grades*, p. 211.
[29] Biot, *Traité de Physique*, p. 70.
[30] Biot, *Traité de Physique*, p. 71: Cavallo, physicien exact et ingénieux, rapporte dans les Transactions philosophiques, qu'ayant accordé soigneusement une harpe ordinaire sur ces principes, en se servant d'un bon monochorde, l' exécution s'y est trouvé très-bonne dans tous les tons et dans tous les modes.

> serving himself with a good monochord, the performance has been found very good in all tones and in all keys.

Actually, Tiberius Cavallo tuned a harpsichord with the help of a monochord:[31]

> In order to hear the effect of the above-mentioned temperament of equal harmony, I had a monochord made in a very accurate manner, and upon it I laid down the divisions for the thirteen notes of an octave properly tempered in the manner explained above. After a great deal of trouble in adjusting the movable fret, correcting the divisions, &c. I at last succeeded so well as to render the divisions exact within at least 300th part of an inch, and every part of the instrument was rendered sufficiently steady and unalterable.
>
> This being done, I had a large harpsichord, with a single unison (in order to judge the better of the effect), tuned very accurately by the help of the monochord. With this instrument, in whatever key the performer played, the harmony was perfectly equal throughout, and the effect was the same as if one played in the key of E natural on a harpsichord tuned in the usual manner.
>
> I shall, therefore, conclude with saying, that when the harpsichord, or organ, &c. is to serve for solo playing, and for a particular sort of music, it is proper to tune in the ususal manner, *viz*. so as to give the greatest effect to those concords which occur more frequently in that sort of music; but that when the instrument is to serve for accompanying other instruments or humen voices, and especially whem modulations and transpositions are to be practised, then it must be tuned according to the temperament of equal harmony, which has been explained in the preceding pages.

Is a monochord easy to work with in musical practice? According to Robert Smith, the well-known eighteenth century scientist, it is:[32]

> As the known method of tuning an instrument by the help of a monochord is easier than any other to less skilful ears, and pretty exact too if the *apparatus* to the monochord be well contrived, it may not amiss to shew the manner of dividing it according to any proposed temperament of the scale.

A short, but interesting discussion of this question has been given by Johann Carl Fischer, in his *Geschichte der Physik*. Fischer mentioned Mendelssohn’s contribution with credit:[33]

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[31] Cavallo, *Of the temperament*, p. 254.
[32] Smith, *Harmonics, second edition*, p. 223.

It has already been brought about in Part IV, page 254, that equal temperament is the way, with which the best possible approximation of perfect consonants is simultaneously achieved. How this equal temperament can geometrically be construed, is demonstrated by Moses Mendelssohn in Marpurg's Historico-critical Contributions to the Reception of Music, in the Second Part of the Fifth Volume.

Fischer mentioned the difficulty of constructing a good monochord, but he added another objection:[34]

Without doubt, this equal temperament is how the best possible approximation of perfect consonants is simultaneously achieved. The whole tones proceed all by the ratio of 8909/10000, which differs very few from 8/9; the fifths and fourths deviate only by a twelfth, and the thirds by a third of a comma, which is equal to the difference of the larger and the smaller whole tone (8/9 : 9/10 = 80/81), what is regarded as the largest deviation of the perfect consonance that is still tolerable for the ear. Nevertheless musicians have found this great difficulty with equal temperament that the tuning is only possible with an precisely divided monochord; in addition they have found it also a disadvatange that all fundamental tones become completely the same. Thereby the valuable advantages would be lost which one could otherwise draw out of the manifold of the characters of the different keys, which no sensitive composer would readily to abandon.

Sir James Jeans remarked in his well-known book *Science & Music*,[35] that already Robert Smith, 'writing in 1759, described equal temperament as "that

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[33] Fischer, *Geschichte der Physik*, p. 563: Es ist schon im Th. IV, S. 254, angeführt worden, daß die gleichschwebende Temperatur diejenige ist, bey welcher die möglichste Annäherung an die Reinigkeit für alle Consonanzen zugleich erhalten wird. Wie diese gleichschwebende Temperatur geometrisch construirt werden könne, hat Moses Mendelssohn in Marpurg's historisch-kritischen Beyträgen zur Aufnahme der Musik, im 2ten Stücke des 5ten Bandes, gezeigt.

[34] Fischer 1805, p. 254: Diese gleichschwebende Temperatur ist nun ohne allem Zweifel diejenige, bei welcher die möglichste Annäherung an die Reinigkeit für alle Consonanzen zugleich erhalten wird. Die ganzen Töne schreiten sämmtlich durch das Verhältniß 8909/10000 fort, welches von 8/9 sehr wenig abweicht; die Quinten und Quarten weichen nur um den zwölften, und die Terzen um die dritten Theil eines Comma ab, welches dem Unterschiede des größern und kleinern Tons (8/9 : 9/10 = 80/81) gleich ist, und für die größte dem Gehör erträgliche Abweichung von der Reinigkeit angenommen wird. Gleichwohl haben die Tonkünstler bey der gleichschwebenden Temperatur diese große Schwierigkeit gefunden, daß die Stimmung nicht anders, als nach einem genau eingetheilten Monochord möglich ist; überdem haben sie auch dieß als ein Nachtheil angeführt, daß in der gleichschwebenden Temperatur alle Grundtöne einander völlig gleich werden, wodurch die schätzbaren Vortheile verloren giengen, welche man sonst aus der Mannichfaltigkeit des Charakters der Tonleitern von verschiedenen Grundtönen ziehe, und welche kein Tonsetzer von Gefühl gern aufopfern werde.

[35] Jeans, *Science and Music*.

inharmonious system of 12 hemitones, which produces a harmony extremely coarse and disagreeable"'.[36]

The quotation is not quite correct, as can be seen from the following fragment,[37] taken from the second edition of *Harmonics*, but the idea is clear: skilled musicians of that time, like Robert Smith himself – he played the harpsichord and gave lessons – did not accept equal temperament.

> Now for want of another sound to terminate each diesis in the scale, it is necessary in the tuning to diminish the diesis till one sound may serve tolerably for the other, and thus to approach towards that inharmonious system of 12 hemitones, till the harmony of the scale becomes very coarse before the false consonances are barely tolerable (...).

In the first edition Smith wrote that '*Euler* (*b*) and others disapprove of incommensurable vibrations as impracticable and inharmonious.' He quoted from *Euler's Tentamen novae Theoriae musicae*, cap. ix.sect. 17. Petropoli. 1739, in the footnote, indicated by *b*.[38]

It was pointed out in the *Introduction* that the performance of eighteen century compositions on organs and harpsichords with the then usual tuning became more and more difficult. Now we see that the radical solution with equal temperament also met with opposition.

Mendelssohn's rudimentary aesthetics of music

Mendelssohn did not have an elaborated aesthetics of music, but some of his remarks on music seem to run counter to his advice on tuning instruments in equal temperament. We saw that Euler definitely preferred harmonic musical intervals, and Mendelssohn paid attention to harmonic proportions of vibrations in Anmerkung *l* of the *Briefe über die Empfindungen*. It seems that Mendelssohn saw a direct connection of harmonic vibrations and the movements inside the ear. In the *Rhapsodie oder Zusätze zu den Briefen über die Empfindungen* he stipulates that composers must embellish the tones of nature (*Töne der Natur*). Again he remarks that these natural signs produce effects on the organs of hearing. He distinguishes melody and harmony in this respect.

However, the tone relations of equal temperament have nothing 'natural', with the exception of octaves. Therefore it is difficult to imagine that Mendelssohn would have incorporated equal temperament in an elaborated aesthetical theory of music. It is also difficult to imagine that Mendelssohn would have been a proponent of equal temperament tuning if he had heard its sounding results. It is

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[36] Jeans, *Science and Music,* p. 185.
[37] Smith, *Harmonics, second edition*, p. 167.
[38] Smith, *Harmonics*, p. 124-125.

more plausible to assume that he would be ‘shocked by the wide thirds’.[39] But this is, of course, speculation. We simply don’t know if Mendelssohn tuned an harpsichord at all according to his own prescription. His teacher Kirnberger was certainly not a proponent of equal temperament.

Negative appreciations of equally tempered instruments persisted throughout the nineteenth century. A serious French author saw such damaging effects on singers who get accustomed to studying at the piano, that he wanted to ban all tempered instruments from singing schools.[40] Obviously he had read Helmholtz:[41]

> The singer, who practises with the help of a tempered instrument, has no principle at all, with which he can surely and precisely measure the pitch of his voice.

It is well-known that Helmholtz ‘was greatly impressed by the use of the Tonic Sol-fa method of instruction in England, and convinced that choirs trained by this method sang true intervals when unaccompanied.’[42]

This means that the melody of Example 1 should be practiced in the indicated way. See Example 23:

Example 23

What about Mendelssohn’s hearing qualities? Could he hear the difference between the *e’’*s of this melody? Or was he more a mathematician than a musician? Kayserling’s commentary seems to confirm this:[43]

> Without being capable to play an instrument in the true sense of the word , or to rightly hit the tones in singing, he could easily calculate all musical

[39]Approximately the tempered third is higher than the true by the interval 126:125. Rayleigh 1926, p.11.
[40] Langel, *De stem, het oor en de muziek*, p. 146.
[41] Helmholtz, *Die Lehre von den Tonempfindungen*, p. 527: Der Sänger, welcher sich an einem temperirten Instrumente einübt, hat gar kein Princip, nach welchem er die Tonhöhe seiner Stimme sicher und genauer abmessen könnte.
[42] Wood, *The Physics of Music*, p. 194. The remark is based on Helmholtz’s own report: Helmholtz 1877, p. 666.
[43] Kayserling, *Moses Mendelssohn*, p.77; Kayserling, *Moses Mendelssohn*. Zweite Auflage, p. 64-65: Ohne ein Instrument im eigentlichen Sinne des Wortes spielen, oder die Töne im Singen treffen zu können, war er im Stande, alle Verhältnisse in der Musik, die Versetzungen der Accorde, die verschiedenen Combinationen der Töne u. s. w. leicht auszurechnen.

proportions, the transpositions of chords, the different tone combinations etc.

That Mendelssohn was a good mathematician is already obvious from his Euclidean treatise on equal temperament!

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