MATHEMATICAL PUZZLES

(*It is a Saturday afternoon. Math, Log, and Comp are together in Comp's home, after a busy week.*)

COMP. At last, time for fun!

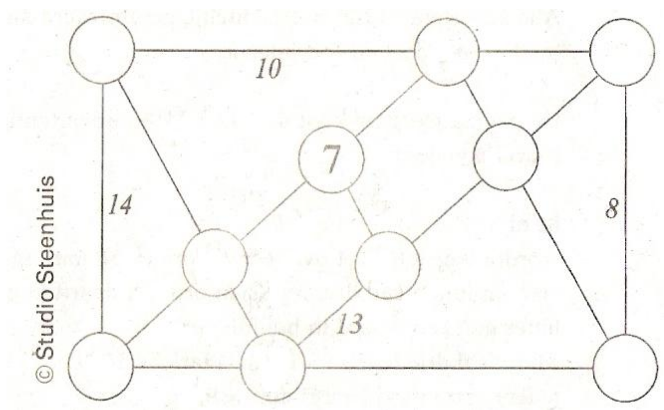
LOG. What are you up to?

COMP. Do you know a funny game or puzzle?

LOG. I am sure that Math has plenty of such things!

COMP. I am afraid that he has only mathematical ones. (*At the same time the daily newspaper*[1] *is delivered. Math takes it out of the mail box.)*

MATH. What do you think of this puzzle? (*He opens a page and shows the following picture.*)



The task is to place the numbers 1 to 10 in the circles such that all sums of two adjacent numbers are different, more precisely, that each of the numbers 3 to 19 occurs exactly once.

LOG. I am not sure if Comp considers this a funny puzzle.

COMP. I have seen more of these problems. They are developed by an expert, Bertil Pouwels in cooperation with Studio Steenhuis. I think that they can be simply solved by a computer program. But Math always wants to solve them by hand, so to say. And I think he likes it. Do you, Math?

[1] *NRC Handelsblad*. The following picture has been reproduced with permission of Studio Steenhuis. I am very grateful for that.

MATH. I am not so much interested in this particular puzzle, because some numbers and sums are already given. Nevertheless the figure is interesting, even beautiful, if I may say so myself.

LOG. I see that it has point symmetry. Would that be reflected in the distribution of the numbers and the sums?

MATH. No doubt it does. It is a mathematical problem, isn't it?

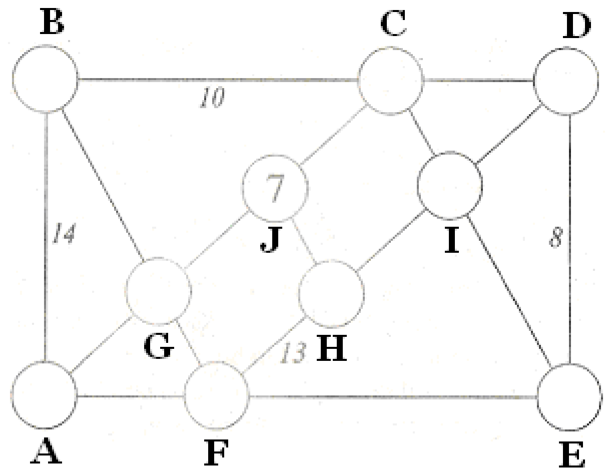
COMP. Apart from this, it can be systematically solved by starting with one of the circles, and creating a search tree.

LOG. That is right, but keep thinking when you are doing it. For example, the sum 3 can only be obtained by the numbers 1 and 2, and the sum 4 by 1 and 3. Similar conclusions hold for 19 and 18. Notice that four numbers occur in four sums (*he points at the circles from which four lines depart*) and the rest in three. And as always there is the problem where to begin.

COMP. I learned in your course on Artificial Intelligence, Math, that a good strategy is to begin with the places which give most information about what follows. I see three such places. (*He points at the circles forming the endpoints of the sums 14 and 10.*) Then I prefer to use a breadth-first search.

MATH. Go ahead, Comp. After that I will use a different approach, reducing the problem to its simplest form with only three points, or even, if you want, one point.

COMP.



A = 4	B = 4	C = 1
B = 10	A = 10	CJ = 8
	C = 6	
	CJ = 13	

A = 5	B = 5	C = 2
B = 9	C = 5	B = 8
C = 1		A = 6
		D = 1
		E = 7

LOG. Stop! If C = 2 and D is not 1, then I must be 1, otherwise we cannot get the sum 3 anymore!

COMP. OK, I will use it:

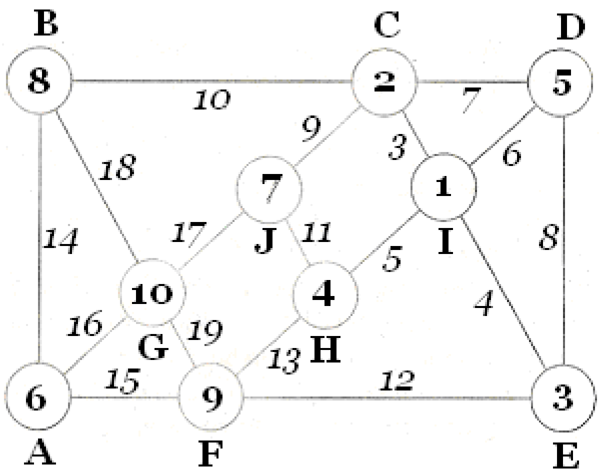
C = 2
B = 8
A = 6
I = 1
CJ = 9
CI = 3

LOG. Wait a minute. Now there are only two possibilities for 3, either D or E in order to get the sum 4.

COMP.

D = 3	E = 3
E = 5	EI = 4
F = 7	D = 5
	DC = 7
	DI = 6
	F = 9
	FA = 15
	H = 4
	HJ = 11

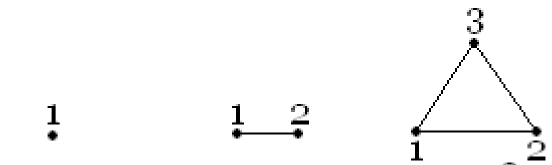
LOG. Then G must be 10, and this solves the problem:



MATH. It is interesting that the point symmetry is reflected in the numbers and in the sums. Each point n has a matching point 11 – n, and it follows that each sum has a matching sum 22. If we had known this before, we would have found the solution straightaway.

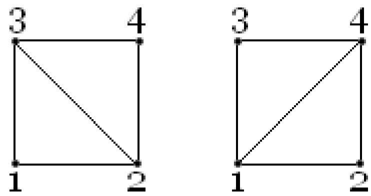
LOG. I assume that you still want to follow your procedure?

MATH. Of course. Although n = 1, n = 2, and n = 3 are straightforward, I will draw their pictures:



$n = 1$ $n = 2$ $n = 3$

LOG. It is clear that the solution for n = 4 consists of 12, 13, 43, 42, and either 23 or 41. In either case we have point symmetry. I avoid intersecting lines:



$n = 4$

MATH. It is clear that these graphs, and hence these solutions are isomorphic. We need only to interchange first 1 and 2, and then 3 and 4. (34)(12) is a notation of this transformation. But let us see what is the case with n = 5. I expect different solutions.

COMP. I will help you with the construction of a search tree:

			12			
			13			
		14		23		
	15		24		15	
25		34		25		34
	35				35	
	45				45	

LOG. I see. So there are eight possibilities:

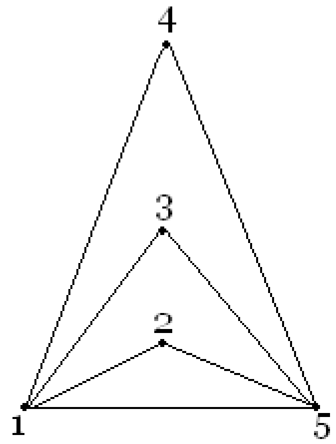
12, 13, 14, 15, 25, 35, 45
12, 13, 14, 15, 34, 35, 45
12, 13, 14, 24, 34, 35, 45
12, 13, 14, 24, 25, 35, 45
12, 13, 23, 24, 34, 35, 45
12, 13, 23, 24, 25, 35, 45
12, 13, 23, 15, 25, 35, 45
12, 13, 23, 15, 34, 35, 45

Some of them are clearly isomorphic. This can be seen by applying the transformation $k \rightarrow 6 - k$ to their elements.

54, 53, 52, 51, 41, 31, 21
54, 53, 52, 51, 32, 31, 21
54, 53, 52, 42, 32, 31, 21
54, 53, 52, 42, 41, 31, 21
54, 53, 43, 42, 33, 31, 21
54, 53, 43, 42, 41, 31, 21
54, 53, 43, 51, 41, 31, 21
54, 53, 43, 51, 32, 31, 21

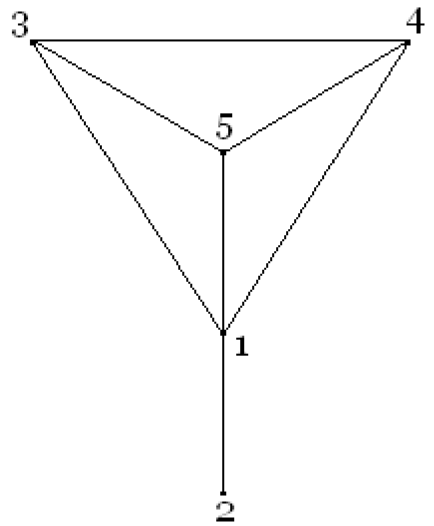
It follows that some of them have the special property that they are point symmetrical. The transformation $k \rightarrow 6 - k$ turns them into themselves.

MATH. This should appear from their figures. That is easy. Take the first one.

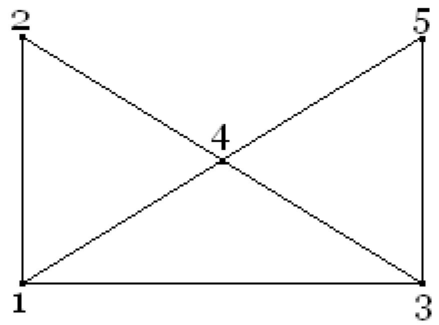


COMP. The points 1 and 5 have a special role, because they form part of four lines, whereas the other points must be content with two.

MATH. The frequency of 2 in the second solution is even smaller, 2 occurs only once. It can directly be seen from the picture:

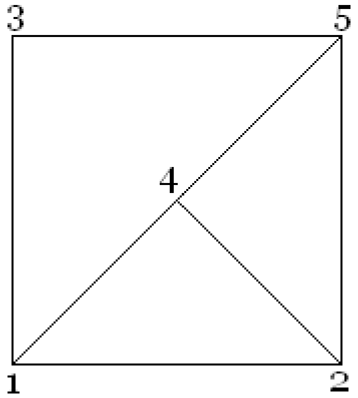


LOG. Three different frequencies occur in the third solution. It is easy to draw the picture:



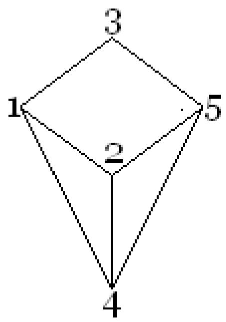
MATH. The fourth solution is an exception. It has only two different frequencies:

n	1	2	3	4	5
f	3	3	2	3	3



12, 13, 14, 24, 25, 35, 45

LOG. I think of another configuration of the same solution:



MATH. Very nice!

COMP. I assume that the number of solutions increases with the number of points. My suggestion is therefore to introduce a constraint: accept only solutions with at most three different frequencies. Shall I try to find such a solution for n = 6, this time with a depth-first search?

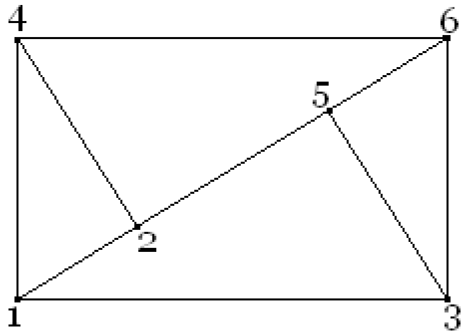
MATH. Do your best, Comp.

COMP.

		12	
		13	
	14		23
15		**24**	
	25		34
26		**35**	
	36		45
	46		
	56		

LOG. We are lucky, all numbers have the frequency 3. Math, do you draw the figure?

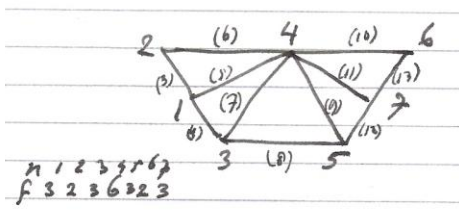
MATH. No problem:

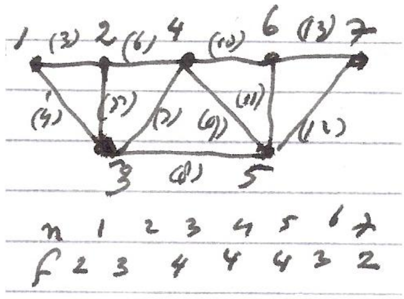


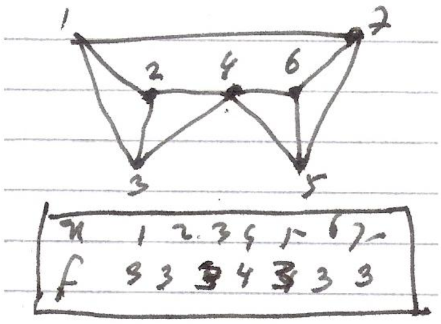
COMP. Is this accidental, that all frequencies are the same?

LOG. It depends on the value of n. The sums are, in general, all numbers from 3 to $2n - 1$, that is $2n - 3$. The formula for the sum of the frequencies is therefore $4n - 6$. If $n = 6$, $4n - 6 = 18$, and this is divisible by 6 itself. If $n = 7$, $4n - 6 = 22$, and then we can hope that there is a solution with six frequencies 3, and one frequency 4.

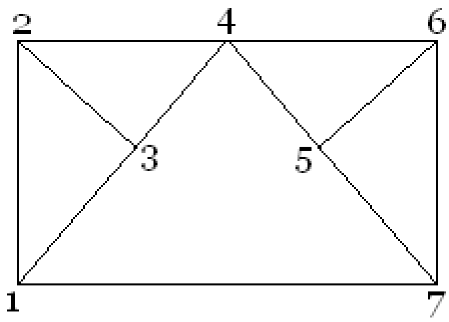
MATH. Wait. (*He tries some figures on a piece of paper:*)





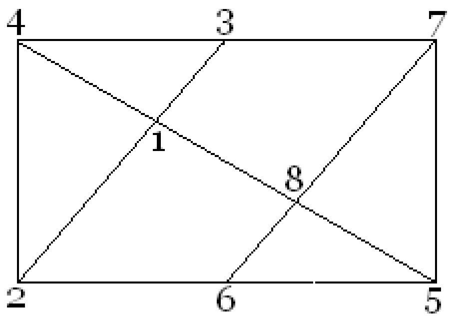


MATH. See what I found:



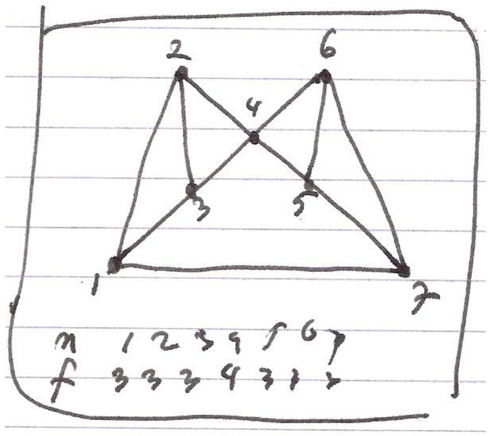
LOG. It seems as if 4 links the solution for 1, 2, 3 with the solution of 5, 6, 7. This offers perspectives for n = 8.

MATH. Indeed:

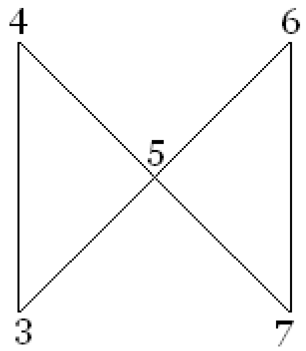


COMP. Do you think that it also works for n = 9?

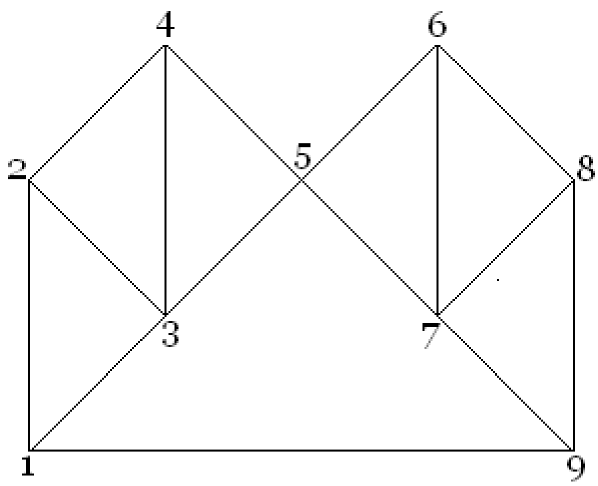
MATH. Look at the figure for n = 7, in the following form: (*He shows his scribble-paper.*)



The middle number 4 is in the center of the figure and it is connected with two figures for n = 3, respectively with the numbers 1, 2, 3, and 5, 6, 7. Notice also the symmetry of the figure: opposite numbers have everywhere the sum 8. This suggests that we start with the middle number 5 of the nine numbers for n = 9, taking into account the symmetry:

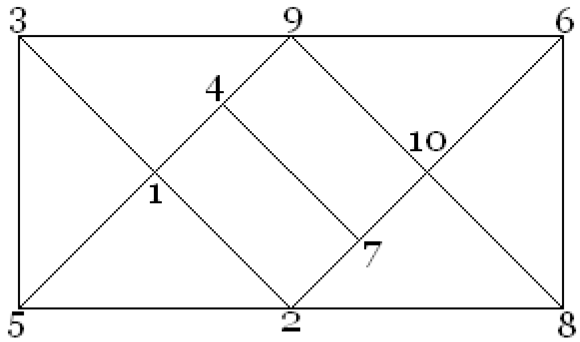


Using also 3, 4, 6, and 7, we already have the sums 7, 8, 9, 11, 12 , 13. But before creating the sum 10, I add the solutions for n = 4, both with 1, 2, 3, 4, and 6, 7, 8, 9:



Notice that I obtained the sum 10 by connecting 1 and 9, otherwise these points would have only two frequencies.

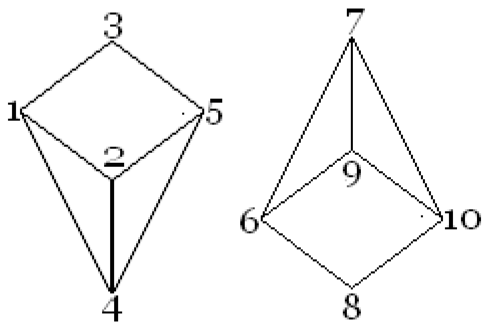
COMP. Very nice! The two solutions for n = 4 stand apart. This reminds me of the solution for n = 10 we started with. Let me take it over:



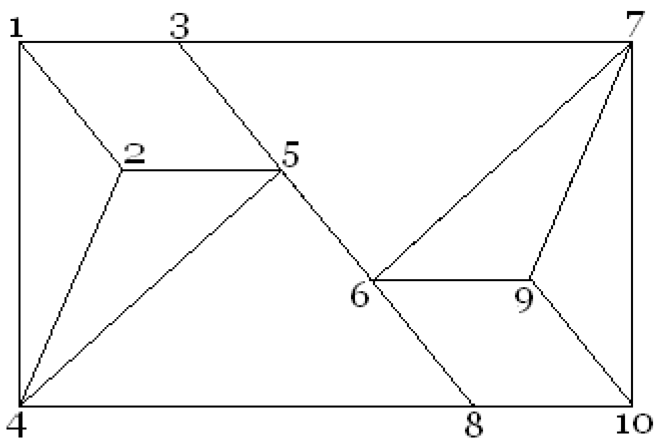
It looks as if two solutions for n = 5 are combined. Is that perhaps a new point of view for the construction of solutions?

MATH. Good idea. Moreover we have already several solutions for n = 5.

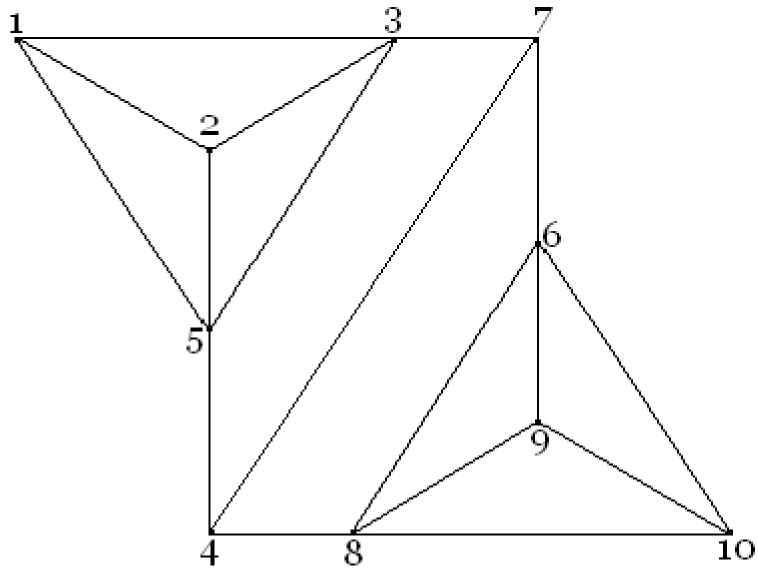
LOG. Especially my favorite one:



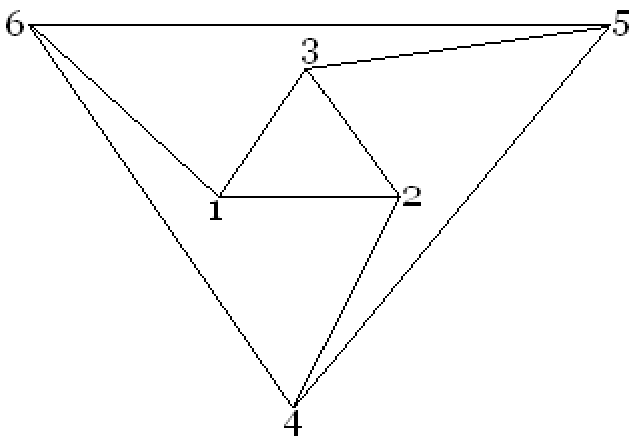
MATH. I see what you are up to:



MATH. And what do you think of this:



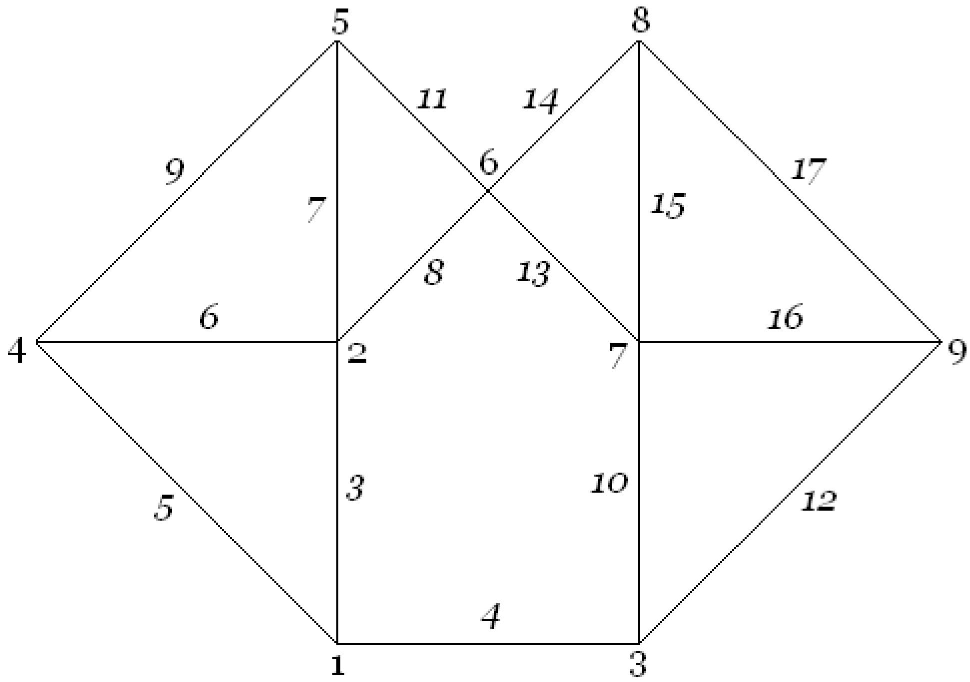
COMP. It is completely clear to me. I can even invent a new configuration for n = 6, look:



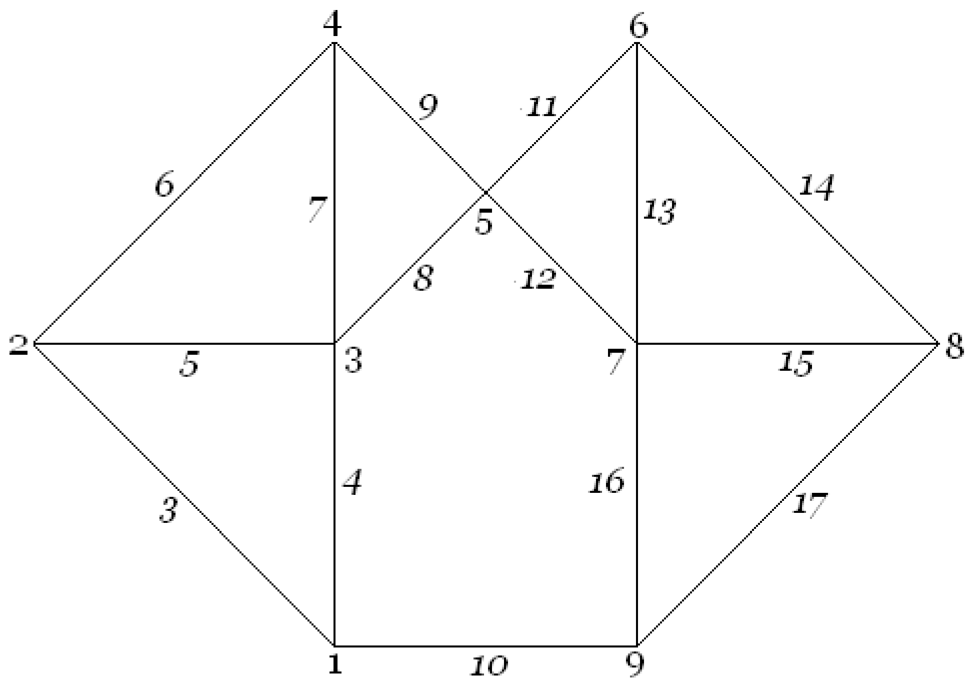
(*All are laughing*.)

LOG. Up till now, we saw, in a sense, regular solutions. But is this necessary? I remember the solution of another puzzle from your excellent newspaper:[2]

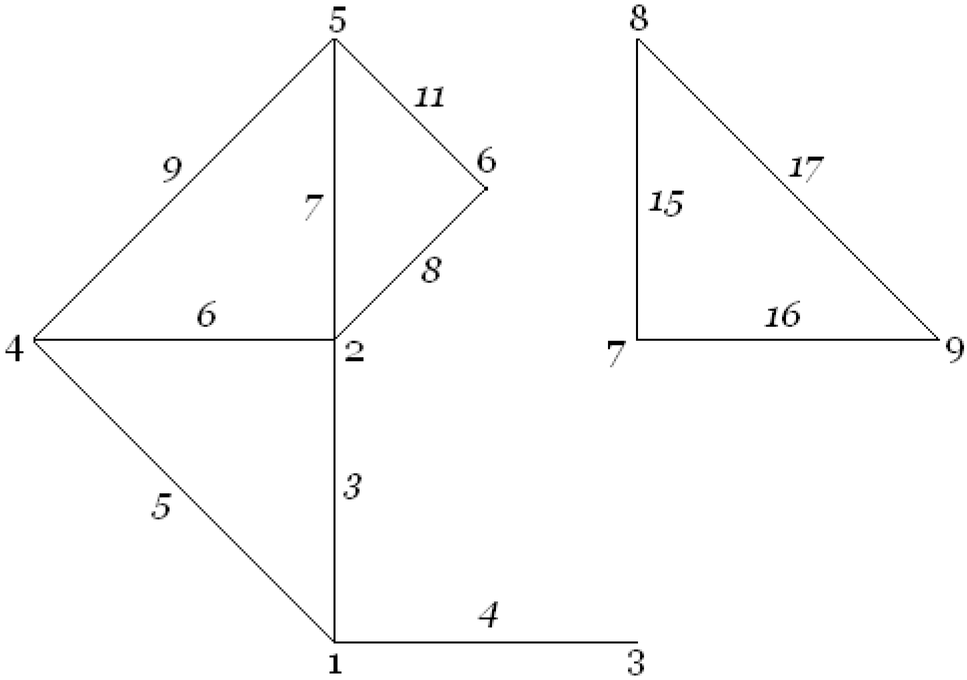
[2] *NRC Handelsblad*, November 8, 2008. By courtesy of Studio Steenhuis.



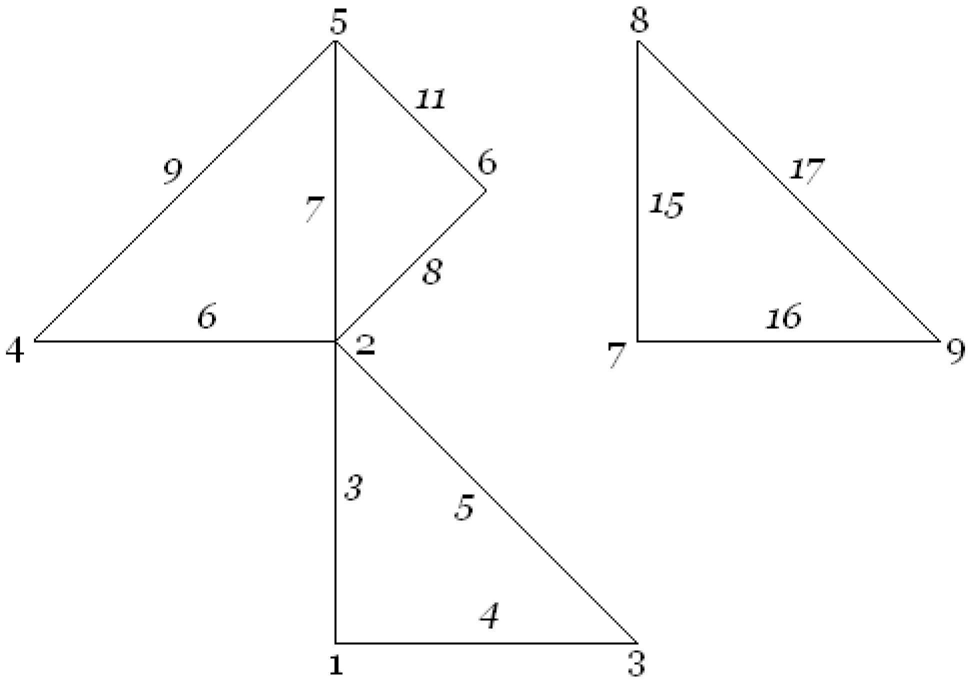
MATH. Very ingenious! It is quite different from my symmetrical solution with the same geometrical figure:



Your solution contains a solution for n = 4, applied to 6, 7, 8, and 9, conbined with a partial solution for n = 6:



With a single translation it contains also a solution for $n = 3$, applied to 1, 2, and 3:

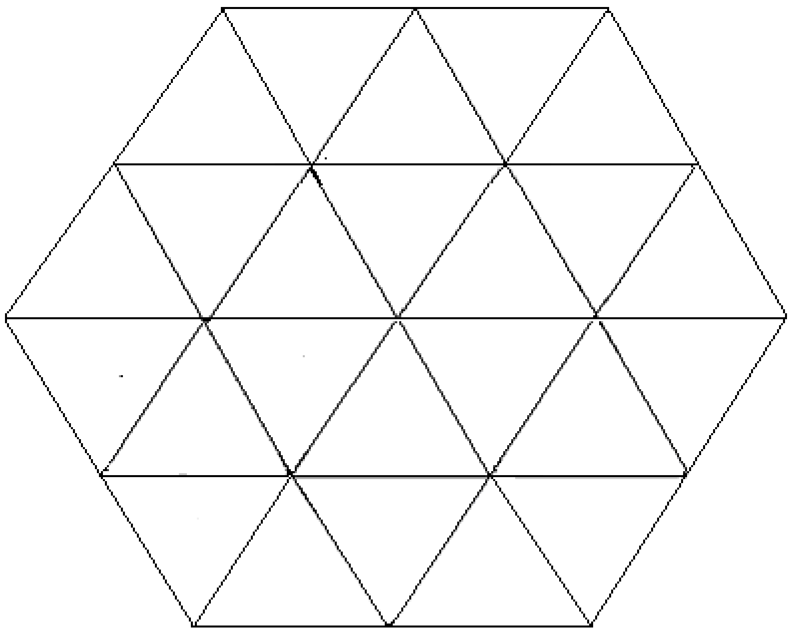


But then I miss a perspicuous solution for the remaining numbers 4, 5, and 6.

COMP. I assume that the makers of the puzzle with the irregular solution found it with the help of a computer program which gave all solutions for $n = 9$.

LOG. As far as I know Math, he is chiefly interested in regular solutions, and only takes recourse to the computer when his so-called ‘intuitions’ fall short.

COMP. Let us see what he can do with his famous intuitions in other puzzles, to be precise, finding figures with equal sums. I do not mean magical squares, but I remember that the same newspaper in which we found the puzzle we started with, also had puzzles with a hexagon divided into triangles. They all had the following form:



The problem was to put numbers into each of the smallest triangles, such that the sum of the numbers in each of the larger triangles consisting of four smallest triangles is everywhere the same. In one of the problems each of the numbers 1 to 8, 8 included, had to be placed in such a way that every number would occur exactly three times. In another problem the numbers 1 to 4, 4 included, had to be placed such that every number would occur six times. In each of such puzzles, some numbers were already placed, in order to facilitate the solution.[3]

MATH. I knew them too, but I wondered why there was no puzzle given in which each of the numbers 1 to 24 had to occur only once. Therefore I tried to solve this problem by first trying to solve such puzzles with simpler figures, and fewer numbers, in the hope that this procedure

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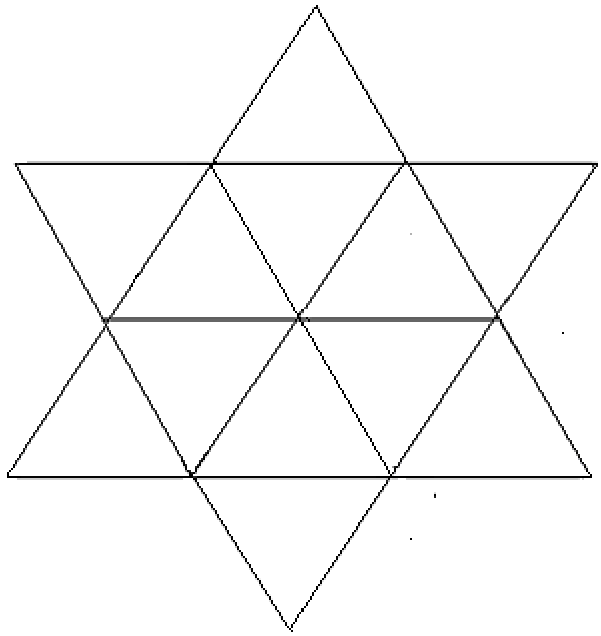
[3] *NRC Handelsblad*. I am grateful that Studio Steenhuis enabled me to use the idea of these problems.

would lead me to a solution of the problem just mentioned. However, I did not get further than a solution for the following figure:



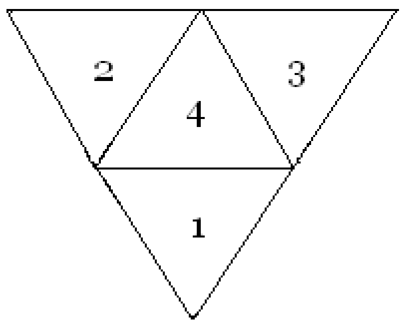
LOG. Can you show how you reached it? Perhaps we can try together to solve your bigger problem.

COMP. I can write a computer program and see if it can directly solve this problem.

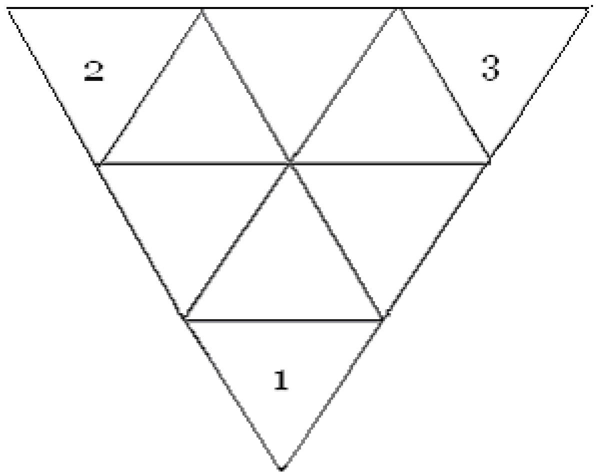
MATH. Good idea! In the mean time I will show LOG how far I have got. (*Comp leaves the living room in order to use his computer in his study.*)

LOG. I assume that you started with the trivial case of one triangle consisting of four smaller ones?

MATH. Sure, and I called it the case of $n = 2$ and $s = 10$.



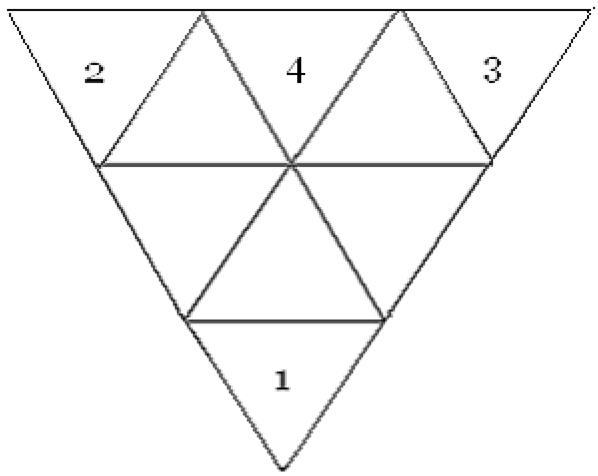
The next case was, of course, n = 9 and s = 20. I solved this problem by first writing out all possible sums for? 20, and subsequently finding out which numbers could be successfully placed in the following figure:



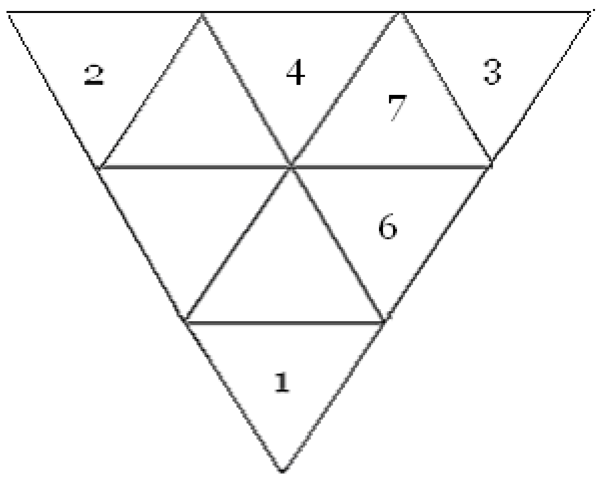
LOG. Let me see if I can do it too:

20 = 1 + 2 + 8 + 9
= 1 + 3 + 7 + 9
= 1 + 4 + 6 + 9
= 1 + 4 + 7 + 8
= 1 + 5 + 6 + 8
= 2 + 3 + 6 + 9
= 2 + 3 + 7 + 8
= 2 + 4 + 5 + 9
= 2 + 4 + 6 + 8
= 2 + 5 + 6 + 7
= 3 + 4 + 5 + 8
= 3 + 4 + 6 + 7

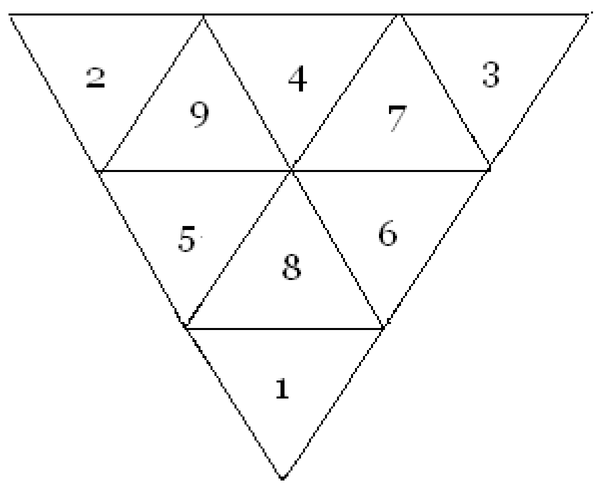
I start with the last sum, and place the 4 in the middle between the 2 and the 3:



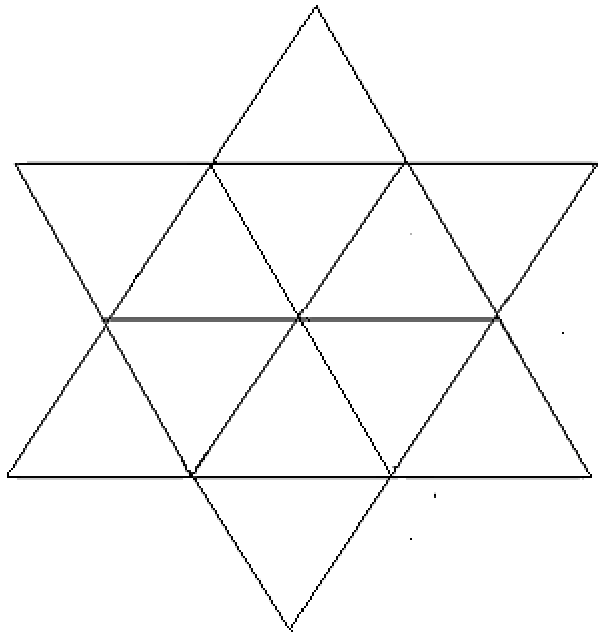
I will see what happens when I place the selected sum such that the 6 comes in the middle between the 3 and the 1:



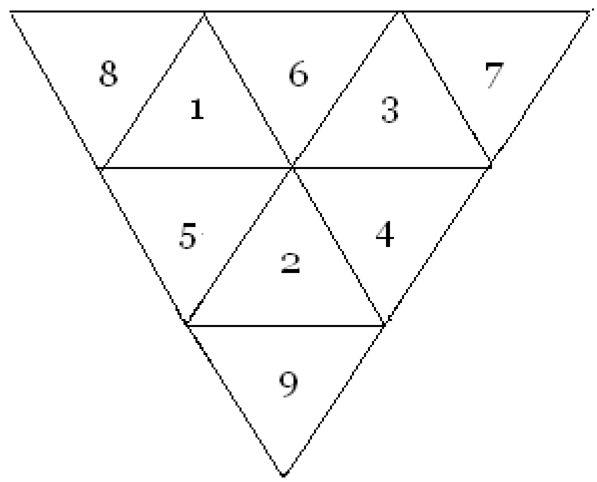
Is it possible to place the 5 in between? the 1 and the 2? Yes, for we can use 1 + 5 + 6 + 8 and 2 + 4 + 5 + 9, look:



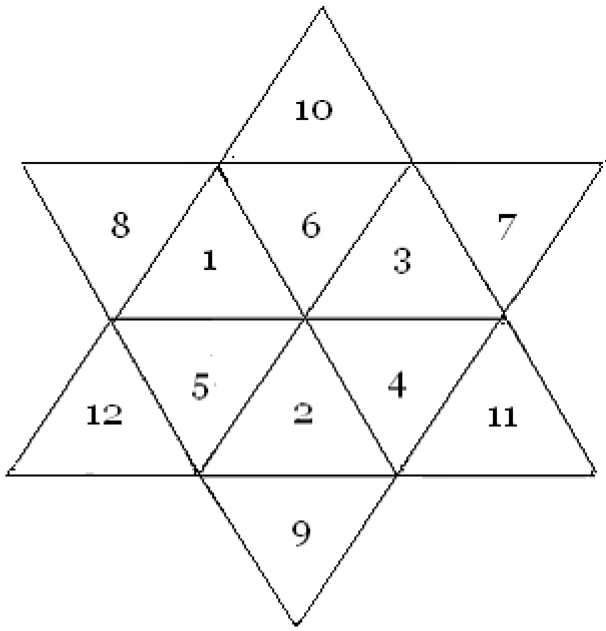
MATH. I don't know any longer how I found the same solution, but I used it to solve the problem for n = 12 and s = 20, that is to say the problem for the star figure:



It is clear that our solution for n = 9 and s = 20 cannot immediately be taken over, because the sum of 9, 4, and 7 is already 20. Therefore I applied the transformation $k \rightarrow 10 - k$ to this solution:

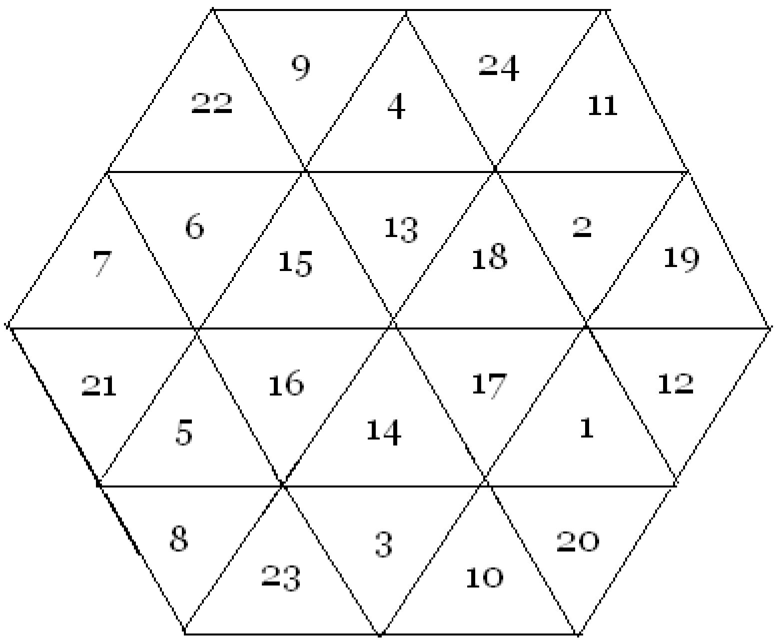


This figure can easily be completed to find a solution for the star figure problem:



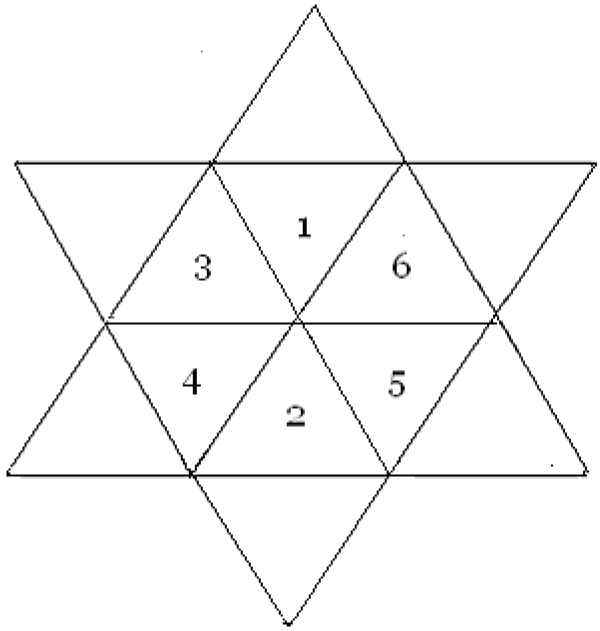
So far so good, but I didn't see how to proceed in order to find a solution for the case of n = 24 and s = 50.

COMP. (*who just entered the room*.) But I did. I even found 206.182 solutions. And what may make you happy, Math, the last ones were extra nice, for example:

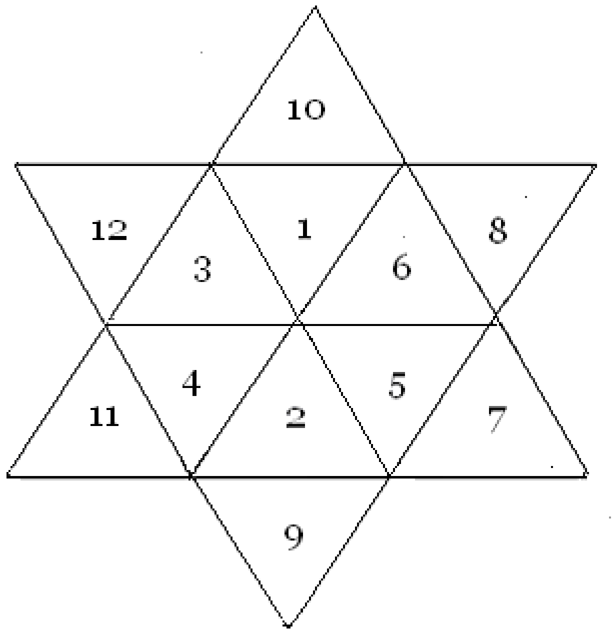


The kernel consists of the numbers 13, 14, 15, 16, 17, 18; they are surrounded by 1,2, 3, 4, 5, and 6. And the numbers 7, 8, 9, 10, 11, 12, and, respectively, 19, 20, 21, 22, 23, 24 occupy the remaining places. Perhaps this particular solution can also be found by hand![4]

MATH. Marvellous! This gives me an idea! To begin with, let us see if we can derive a solution for the star figure from this computer solution! For example, by first applying the transformation $k \to k - 12$ to the inner hexagon, your kernel:



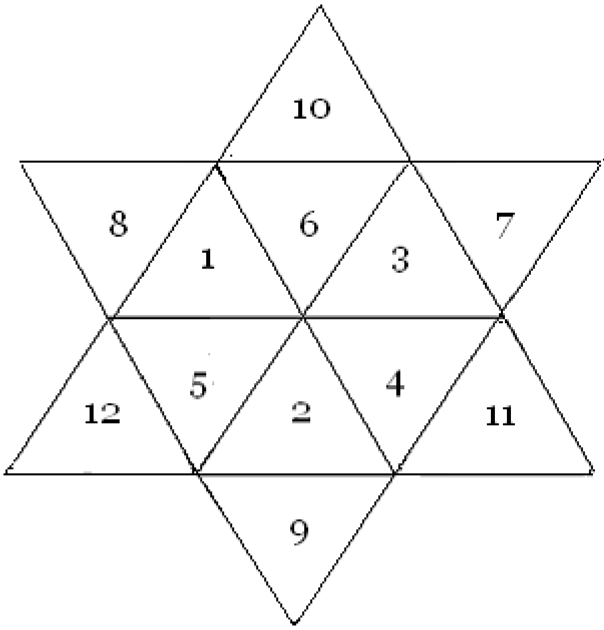
This is already sufficient, for the only thing we have to do is to apply the transformation $k \to k + 6$ to the surrounding numbers.



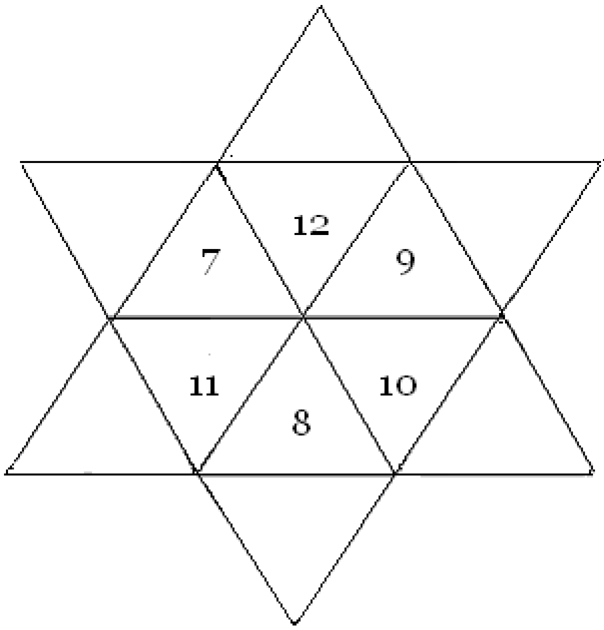
[4] I am very grateful to Dr Jeroen Donkers, to whom these outcomes are due.

LOG. They are also already fixed by the sums.

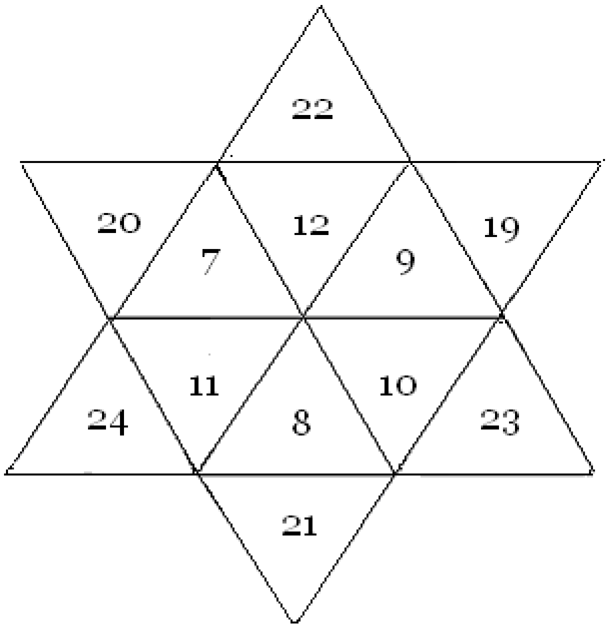
MATH. Of course, but I attach importance to the transformations, because I have the feeling that they can help us to take the other way around! I mean, trying to derive a solution for the case $n = 24$ and $s = 50$ from our own solution for $n = 9$ and $s = 20$:



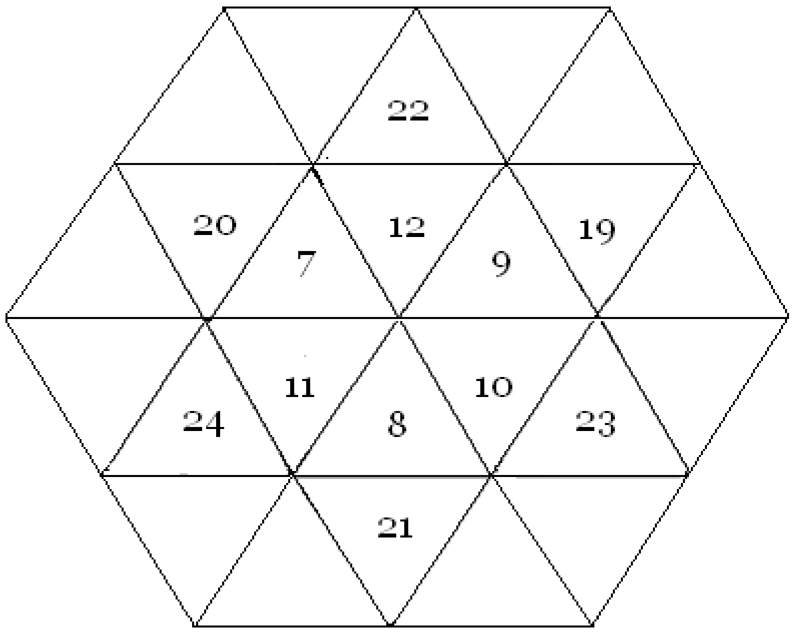
Let me begin with the transformation $k \rightarrow k + 6$ to the inner hexagon:



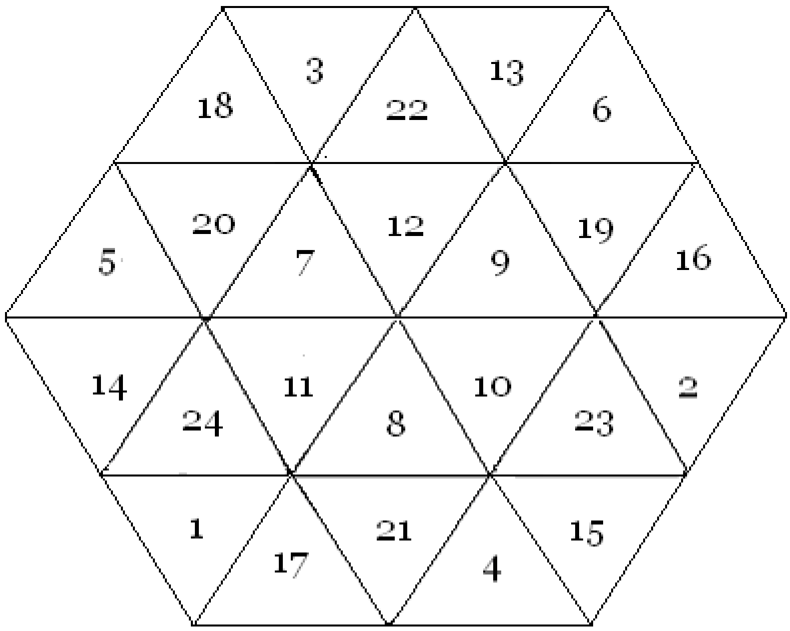
Then the surrounding numbers are fixed; but they are also obtained by applying the transformation $k \rightarrow k + 12$ to the original surrounding numbers:



There remain twelve triangles to be filled:



In order to accomplish this, the numbers 1 through 6 and 13 through 18 are available. We must use two of these numbers to obtain the sums, respectively 50 – (12 + 22), 50 – (9 + 19), 50 – (10 + 23), 50 – (8 + 21), 50 – (11 + 24), 50 – (7 + 20), in other words 16, 22, 17, 21, 15, 23, or also 15, 16, 17, 21, 22, 23. If we take 15 = 1 + 14, then we get 16 = 3 + 13 and 17 = 2 + 15; when, moreover, 21 = 4 + 17, then 22 = 6 + 16 and 23 = 5 + 18. These results can easily be placed:

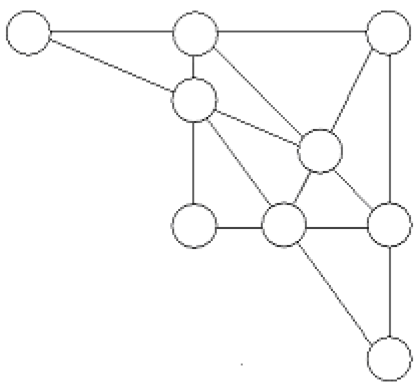


This means that the problem of deriving a solution for the case n = 24 and s = 50 from an already known solution for n = 9 and s = 20 has been solved! But honour to whom honour is due, Comp!

LOG. I think that we must call you in more often, Comp!

COMP. You are welcome! But let me first invite you for a drink!

Although Math, Log, and Comp didn't stop talking about mathematical puzzles, the rest of their conversation is not reproduced. The discussion concerned the possibility of 'equal sums' of three numbers instead of four, excluding cases in which the frequency of a number is greater than three, or in other words, cases in which a point belongs to more than three lines, as is the case with a 'magical square' of three by three. The result was a simple puzzle, reproduced below:



The task is to place the numbers 1 to 9 in the circles such that all sums of two adjacent numbers are 14.