MATHEMATICAL BEAUTY

(*Math, Log, and Comp have just visited an exposition of mathematical art, and back in the L. E. J. Brouwer Institute they have a new subject for discussion.*)

COMP. I liked the computer art at the exhibition, what is your opinion?

LOG. I was impressed by the beauty of the earlier computer drawings by Csuri. The later ones, in color, became less and less mathematical, and I didn't appreciate these examples of computer art.

COMP. Is there something like mathematical beauty in computer art? What do you think, Math?

MATH. As always, beauty is in the eye of the beholder. But the real question is whether an artefact can have certain mathematical properties which evoke aesthetic emotions in some observers.

COMP. What kind of artefacts do you mean? I can understand that artefacts that count as mathematical can have such properties. For example, mathematical principles, mathematical procedures, mathematical proofs, mathematical figures, mathematical formulas, mathematical models, mathematical puzzles, and mathematical games.

LOG. Can you mention mathematical games?

COMP. Draughts, chess, poker, billiards … You can easily amplify my list!

MATH. Billiards is not a mathematical game, simply because it cannot be described in mathematical terms only. There are physical constraints.

LOG. This condition makes it also difficult to speak of mathematical art.

COMP. I accept that. But it does not exclude that a piece of art can have mathematical properties which are considered beautiful by some people, including Log and myself.

MATH. Apparently the question is in general: with which properties of mathematical artefacts does this occur? But I don't think that it is appropriate to try to answer such a general problem. It is better to first inspect special examples of beautiful pieces of mathematics, and see what we can infer from the results of the inspection.

LOG. Then you must give such examples!

MATH. I can only speak for myself, but who knows, do you also regard my examples beautiful.

COMP. Go on, Math!

MATH. Comp mentioned already various kinds of mathematical artefacts, such as mathematical formulas. Let me give a well-known example, Heron's formula. Given is a triangle *ABC*, with lengths of its sides *a*, *b*, *c*.

If we put (*Math uses the blackboard*):

$$s = \tfrac{1}{2}(a + b + c)$$

then the area of *ABC* is

$$\sqrt{s(s - a)(s - b)(s - c)}$$

COMP. This formula is new to me!

LOG. You belong to the generation which did not learn geometry in the way Math and I did. But what do you think of this formula?

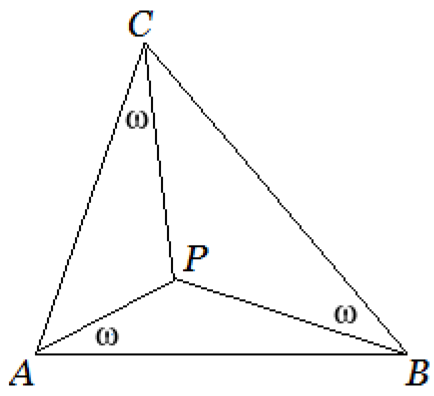
COMP. Very interesting, even beautiful, especially because the formula for *s* is also quite illuminating, certainly if we write it slightly different:

$$\frac{a + b + c}{2}$$

MATH. So what makes it beautiful to you?

COMP. Its simplicity.

MATH. Geometry, as I learned it in my youth, is full of simple formulas. Since Antiquity mathematicians were looking for beautiful theorems, in the hope to get immortality. Look at the following figure:



Brocard proved that there is exactly one point *P* with the property that the three marked angles ω are equal. But this was not all. If the angles of the triangle *ABC* are α, β, and γ, then the following equation applies:

$$\cot \omega = \cot \alpha + \cot \beta + \cot \gamma$$

Since then, the point *P* is called a Brocard point.

COMP. Long live Brocard!

LOG. It seems that geometricians have a preference for formulas of a certain kind. But what do you think of

$$e^{i\pi} + 1 = 0$$

COMP. That is more familiar to me. It is easy to prove it with the help of Euler's formula

$$e^{ix} = \cos x + i\sin x$$

I remember how surprised I was when I solved the problem of differentiating the function

$$\frac{\cos x + i\sin x}{e^{ix}}$$

MATH. There is a story that tells what Richard Feynman wrote in his diary, when he, fourteen years old, saw this formula for the first time: 'the most remarkable formula in math'. Both formulas are remarkable indeed, the former because it brings the most famous mathematical constants, 0, 1, π, e, and *i* together in one short equation. But who finds it therefore beautiful?

LOG. To me, this is a rhetorical question. Simplicity is not a necessary condition for beauty.

MATH. We need more examples. What do you think of the next one:

$$\int_0^\infty \frac{\sin y}{y}\, dy = \frac{\pi}{2}$$

LOG. Many Fresnel integrals have nice properties. Here is another one:

$$\int_0^\infty \frac{\sin y}{\sqrt{y}}\, dy = \sqrt{\frac{\pi}{2}}$$

MATH. I am glad that you also mentioned the variant with the square root sign. We must consider the two integrals together, in order to see that adding a square root sign in the left part, results in adding a square root in the right part too. I find this a beautiful property of these integrals.

COMP. It seems that you don't find the formula beautiful, but a dynamic property of the formula so to say.

MATH. To take over your way of speaking, a formula alone is too static to me. Another example Once I asked my students to rewrite the following expression als a single product:

$$(ac - bd)^2 + (bc + ad)^2$$

Not all of them solved this problem, but even the students who failed, found the solution beautiful:

$$(a^2 + b^2)(c^2 + d^2)$$

LOG. Apparently they had not expected this outcome.

COMP. I assume that the outcome was also a surprise for the students who solved the problem. Its simplicity and perspicuity is perhaps not enough for an aesthetical reaction. Suppose, namely, that you would have given the following exercise:

Prove: $(ac - bd)^2 + (bc + ad)^2 = (a^2 + b^2)(c^2 + d^2)$

Then this easy problem would not evoke aesthetical feelings, I think. The same holds, if this formula is presented as a theorem. Moreover the left part of the equation is not very attractive.

MATH. This can be remedied:

$$(ab + cd)^2 - (bc + da)^2 = (a^2 - c^2)(b^2 - d^2)$$

COMP. That is better! You see, I like formulas with a perspicuous structure. Now there is a cyclical progress in the left part of the equation.

MATH. We see that perspicuous structures are not given, but must be made by the mathematicians themselves.

LOG. Do you think that results are passed over, when no perspicuous representation can be found?

MATH. That is too strong, but it is clear that all those simple notations, think of gamma and beta functions are invented in order to get surveyable theorems.

COMP. I am sorry, but I have never heard of such functions.

MATH. I can easily give you the definitions, they are not all to complicated for mathematicians who often work with them. The following equation may suffice to illustrate my point:

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

LOG. It also happens that one gets an unexpected result. In that case it comes as a surprise. I assume that the mathematician who found the theorem $e^{i\pi} + 1 = 0$ was delighted!

MATH. The same holds for practically all mathematics students, when they see this formula for the first time, I think.

COMP. I already told you about my experience with the derivation of the formula via the differentiation of a function which appeared to be constant.

LOG. You used the word ‘process’ in a formula. This reminds me of Pascal’s triangle in which there is also a progress, this time in the way the triangle is built.

COMP. I know, it works with sums of two adjacent numbers:

1
1 2 1
1 3 3 1
1 4 6 4 1

MATH. Do you also know the triangle with which we can compute the subsequent possible ways in which a number of so-called alternatives can be ordered by an irreflexive and transitive preference relation and a transitive indifference relation of strict weak orderings?

COMP. Can you first give examples?

MATH. Yes, suppose we have two alternatives *a* and *b*, then we can prefer *a* above *b*, or *b* above *a*, or we can be indifferent about *a* and *b*, in notation:

(*ab*); *ab, ba*

You see, there are three possibilities.

With three alternatives *a*, *b*, and *c*, there are thirteen possibilities:

(*abc*); *c*(*ab*), (*ab*)*c*, *b*(*ac*), (*ac*)*b*, *a*(*bc*), (*bc*)*a*; *abc*, *acb*, *bac*, *bca*, *cab*, *cba*

It takes some time, but with four alternatives we find seventy-five possibilities. Now we put 1 at the top of the triangle, and multiplicate it first by 1 and then by 2, so as to get the second row, consisting of 1 and 2, together three. Then we multiplicate the 1 by 1, the sum of the 1 and the 2 by 2, and finally the 2 by 3. This results in the third row, consisting of 1, 6, and 6, together thirteen. Similarly, with the factors 1, 2,3, and 4, we get the fourth row, consisting of 1, 14, 36, and 24, together seventy-five.

1
1.1 2.1
1 2
1.1 2.(1 + 2) 3.2
1 6 6
1.1 2.(1 + 6) 3.(6 + 6) 4.6
1 14 36 24

1
1 2
1 6 6
1 14 36 24

COMP. Let me compute the next number of possibilities:

1 14 36 24
1 30 150 240 120

1, 30, 150, 240, 120 make together 541.

LOG. The triangle does not struck me as very special, but the procedure is perspicuous, and, in a sense even beautiful. Am I right whem I suppose that your proof follows the construction? Is it also beautiful?

MATH. The proof leads to the construction, but I hesitate calling it beautiful. Perhaps we can discuss the question of beautiful proofs later on?

(*The dean of the department, a professor in applied mathematics, enters the room.)*

MATH. Good afternoon App, what brings you here?

APP. Nothing special. I was interested in your activities. Sometimes I have the impression that you are engaged in psychological if not philosophical discussions.

MATH. That is not wholly beside the truth. Today, we are talking about mathematical beauty, and we try to find examples in order to get more grip on it. Do you have favourite examples?

APP. I am almost ashamed of saying it, but I am still astonished of the beauty of Pythagoras' theorem. It is amazing that this extremely simple theorem has so many applications. Moreover I think also of surprising proofs, to begin with Euclid's own one.

MATH. We discussed this proof when we dealt with redescriptions, although not in connection with its possible beauty. But talking about the abundant amount of applications of the Pythagorean theorem, I have a new one that was unexpected to me, and I cannot help that I find it beautiful. Undoubtedly Log and Comp remember my preoccupation with triangular numbers. My new example is the problem of finding two triangular numbers which have again a triangular number as their sum. A few days ago I felt dull, and therefore I made by head a list of the first twenty triangular numbers, in order to make a list of such lucky pairs. Let me write them down:

1, 3, 6, 10, 15,
21, 28, 36, 45, 55,

66, 78, 91, 105, 120,
136, 153, 171, 190, 210

COMP. And you noticed immediately that 6 + 15 = 21, in other words that t(3) + t(5) = t(6), I suppose.

MATH. Yes, this is a solution beginning with t(3), and I called (3, 5, 6) an 'Orphic triple', named after the Orphics, who formed a bond of a kind with the Pythagoreans. Theodor Gomperz once called Orphism and Pythagorism *die weibliche und männliche Verkörperung derselben Grundrichtung*, the female and male embodiment of the same fundamental school. Isn't that beautiful?

I went further with t(4), giving 10 + 45 = 55, and I made a list:

(3, 5, 6) ε O
(4, 9, 10) ε O   my new notation for t(4) + t(9) = t(10)
(5, 6, 8) ε O
(6, 9, 11) ε O
(8, 10, 13) ε O
(9, 13, 16) ε O
(11, 14, 18) ε O

Do you also see what I found? Let me slightly rearrange the list:

(3, 5, 6) ε O

(4, 9, 10) ε O

(5, 6, 8) ε O

(6, 9, 11) ε O

(8, 10, 13) ε O

(9, 13, 16) ε O

(11, 14, 18) ε O

LOG. The Orphic triples in one and the same row increase with 3, 4, and 5. Does this have something to do with Pythagoras?

MATH. You are right! (3, 4, 5) is a Pythagorean triple:

(3, 4, 5) ε P my new notation for $3^2 + 4^2 = 5^2$

COMP. What about the lonely Orphic triple (4, 9, 10)?

MATH. Do you remember the Pythagorean triple with whole numbers that comes after (3, 4, 5)?

COMP. My geometrical knowledge is deficient, but I can still answer your question. It is (5, 12, 13).

MATH. Excellent! So I can amplify my list:

(3, 5, 6) ε O

(4, 9, 10) ε O

(5, 6, 8) ε O

(6, 9, 11) ε O

(8, 10, 13) ε O

(9, 13, 16) ε O

(9, 21, 23) ε O

(11, 14, 18) ε O

(12, 17, 21) ε O

(14, 18, 23) ε O

(14, 33, 36) ε O

LOG. I assume that this correct. Is there also a list with the Pythagorean triple (8, 15, 17)?

MATH. Indeed:

(3, 5, 6) ε O
(11, 20, 23) ε O
(19, 35, 40) ε O
(27, 50, 57) ε O

COMP. It is easy to see that (11, 20, 23) and (19, 35, 40) are Orphic triples, 66 plus 210 is 12 times 23, namely 276, and 190 plus 630 is 820, but I must use my calculator to check if (27, 40, 57) is also an Orphic triple.

LOG. Wait (*She takes pencil and paper*) … I got twice the outcome 1653. It's OK. How do you find such solutions? Can you demonstrate it for the Pythagorean triple (7, 24, 25)?

MATH. I look at Orphic triples of the form (m, n, n + 1). Because t(n + 1) – t(n) = n + 1, I consider numbers m such that t(m) = n + 1, to begin with t(5) = 15. This gives

(5, 14, 15) ε O
(7, 24, 25) ε P
(12, 38, 40) ∉ O, because t(40) – t(38) = 40 + 39, and t(12) = 78

Therefore I try t(6) = 21. This gives

(6, 20, 21) ε O
(7, 24, 25) ε P
(13, 44, 46) ε O, because t(46) – t(44) = 46 + 45, and t(13) = 91

Now we can immediately calculate the next Orphic triple. We only need to expand (13, 44, 46) by (7, 24, 25):

(20, 68, 71) ε O

COMP. That is correct; t(20) = 210, and t(71) – t(68) = 71 + 70 + 69, and that is the same as 210.

LOG. No doubt there is a general theorem. Let me try:

if

(p, q, r) ε O and (a, b, c) ε P

implies

($p + a$, $q + b$, $r + c$) ε O

then

$$(p + na, q + nb, r + nc) \in O$$

COMP. Can you say it in words?

MATH. If an Orphic triple can be expanded by a Pythagorean triple so that the result is again an Orphic triple, then this expansion can always be repeated with Orphic triples as the outcomes.

COMP. This sounds as if the proof is a question of mathematical induction.

MATH. Of course. You can give it to your students as an exercise. I give a hint:

Lemma

if

$$(p, q, r) \in O \text{ and } (a, b, c) \in P$$

implies

$$(p + a, q + b, r + c) \in O$$

then

$$(2p + 1)a + (2q + 1)b = (2q + 1)c$$

APP. An interesting equation, certainly when you rewrite it as

$$((2p + 1)a, (2q + 1)b, (2q + 1)c) \in A$$

I like such simple relationships, and verifications which are not immediately obvious give me always a very good feeling.

LOG. What?

APP. Take Math's example of (3, 5, 6) ε O, (3, 4, 5) ε P, and (6, 9, 11) ε O. According to the formula, (6 + 1)3 + (10 + 1)4 must be (12 + 1)5, that is 65, and 21 + 44 = 65 indeed. Of course I didn't expect it otherwise, but nevertheless ...

LOG. You mean, it comes still as a surprise. It seems to me that Math's general theorem, how elementary as it is, is again beautiful because of its surprise character. And I can now understand why mathematicians find so much beauty in results as Desargues' theorem, and Pascal's theorem, cumulating in the conclusion that certain points lie on one and the same line. This can be easily made perspicuous by a concrete figure on a piece of paper or on the blackboard.

MATH. A well-known interpretation of the word ‘concrete’ is: ‘being in time and space’, and this brings me to Comp’s remark about a possible dynamical character. In these geometrical theorems, the surprising conclusion comes after a process of drawing points and lines, and in the beginning there is no indication of the direction in which it will go.

COMP. *Mutatis mutandis,* Math’s exposition of his elementary theorem about Orphic triples has the same property. Who expected that Pythagorean triples would play an essential role in the development of Orphic triples?

LOG. Following the course of a proof or a derivation takes always time. The theorem itself does not contain any reference to temporal properties. But take Pascal’s theorem for a conic sections. I agree that the exposition of this theorem has a dynamical character. In order to explain what this theorem implies, one has first to draw attention to the six points on the conic section, and then to show how the three intersection points have arisen, and finally to pronounce that these points lie on one and the same line. In any interesting if-then theorem the consequence does not immediately follow from the antecedent, and therefore it can remain remarkable, when the antecedent has been exposed, and the consequence is formulated or is made perspicuous by means of a concrete picture.

MATH. An analogous story can be told for equations.

APP. So my $a^2 + b^2 = c^2$ was not that bad?

MATH. Can you work it out?

APP. When I first saw the proof of Pythagoras’ theorem with the help of the complex plane by two applications of Euler’s formula,

$$a + b\mathrm{i} = c\mathrm{e}^{\mathrm{i}\varphi}$$
$$a - b\mathrm{i} = c\mathrm{e}^{-i\varphi}$$

I was a bit angry with myself why I had never thought of factorizing $a^2 + b^2$ into $(a + b\mathrm{i})(a - b\mathrm{i})$:

$$a^2 + b^2 = (a + b\mathrm{i})(a - b\mathrm{i})$$

But I also admired the coherence of different mathematical sectors.

LOG. Don't presuppose the polar coordinates in the complex plane Pythagoras' theorem?

MATH. That is not necessary, but presupposed is the theory of similarity of triangles. It is the same with the well-known proof in which the perpendicular from C divides the right triangle ABC in two similar triangles to ABC. But App rightly emphasizes the coherence of mathematical theories.

COMP. It is remarkable that you brought Euler to the fore, App, whereas you was not yet present when we discussed the beauty of his formula.

APP. If there is any notion that dominates my work,then is it Euler's constant. We work always with exponential functions. And of course, I learned Euler's formula as a student.

MATH. I know. But is there any beauty in the formulas you work with? Even the formula of a simple function such as

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

does not satisfy my conditions for beauty. Applications of mathematics did no good to aesthetics, so to speak. Time has gone that text books and collections of exercises contained almost exclusively beautiful outcomes.

APP. Which books do you mean?

MATH. For example, Hardy's *Course of Pure Mathematics*, but also earlier works such as the *Sammlung der Aufgaben des Aufgaben-Repertoriums* from 1898. The latter book is especially interesting for the examples of analogies between algebraic and goniometric equations:

$$(a + b)(a - b) = a^2 - b^2$$
$$\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$$

$$a(b - c) + b(c - a) + c(a - b) = 0$$
$$\sin\alpha\sin(\beta - \gamma) + \sin\beta\sin(\gamma - \alpha) + \sin\gamma\sin(\alpha - \beta) = 0$$

APP. These equations are new to me. I agree that it is important to give such formulas as proof exercises in order to stimulate mathematically gifted students.

MATH. I think that would be good for all students. It is clear that there are already beautiful formulas on a very elementary level.

LOG. Would it not be better to collect examples of beautiful ànd ugly formulas and proofs in order to get more grip on the rather vague conditions?

MATH. Isn't that a waste of time? We are naive mathematicians, not philosophers. It was nice discussing mathematical beauty, but we can do better and doing real mathematics.

APP. The doing itself is beautiful!

MATH. Just one last remark. I wondered if my example with the Orphic and Pythagorean triples could be generalized, or at least, extended. Suppose that I define Babylonian numbers as follows:

$$\mathrm{b}(n) = n(n + 1)(n + 2) : 6$$

in order to get the sequence 1, 4, 10, 20, 35, 56, ..., and

$$(x, y, z, t) \in \mathrm{B} \;:=\; \mathrm{b}(x) + \mathrm{b}(y) + \mathrm{b}(z) = \mathrm{b}(t)$$

For example, (1, 4, 5, 6) ε B, because 1 + 20 + 35 = 56

Now I use the following definition of cubic quadruples

$$(x, y, z, t) \in \mathrm{C} := x^3 + y^3 + z^3 = t^3$$

Perhaps you are familiar with this example:

$$(3, 4, 5, 6) \in \mathrm{C}$$

COMP. Of course, this is Ramanujan's equation:

$$3^3 + 4^3 + 5^3 = 6^3$$

MATH. Here are some more:

$$(1, 6, 8, 9) \in C$$
$$(3, 10, 18, 19) \in C$$
$$(7, 14, 17, 20) \in C$$
$$(2, 17, 40, 41) \in C$$

LOG. I understand where you were going up to. You imagined that there is a theorem that says

if

$$(p, q, r, s) \in B \text{ and } (a, b, c, d) \in C$$

implies

$$(p + a, q + b, r + c, s + d) \in B$$

then

$$(p + na, q + nb, r + nc, s + nd) \in B$$

MATH. Yes, and I think that the very idea is beautiful! It illustrates my point that mathematicians can also be guided by beautiful ideas, although their dreams do not always come true. In this case the conditions are not sufficient. We have to add one more condition:

$$(p + 1)a^2 + (q + 1)b^2 + (r + 1)c^2 = (s + 1)d^2$$

COMP. Ah! A formula with a perspicuous structure!

APP. All well and good, but now we have already four conditions. Is there any chance that they can be simultaneously fulfilled? If not, then it is an illusion, and an example that you pure mathematicians live sometimes too much in the clouds with your dreams.

COMP. Then it is my task to bring them down to earth! A good deal of work on hand! I will let you know what the outcome is. See you later!

(*Hereby the dicussion ends. Math, Log, Comp, and App retake their normal activities.*)

After some time, Math received the following e-mail message from Comp:

Hallo Math,

I tested all smaller solutions of *a*, *b*, *c*, and *d*, given by Wolfram, to wit:

[3,4,5,6], [1,6,8,9], [3,10,18,19], [7,14,17,20], [4,17,22,25], [18,19,21,28], [11,15,27,29], [2,17,40,41], [6,32,33,41], [16,23,41,44], [3,36,37,46], [27,30,37,46], [29,34,44,53], [12,19,53,54], [15,42,49,58], [22,51,54,67], [36,38,61,69], [7,54,57,70], [14,23,70,71], [34,39,65,72], [38,43,66,75], [31,33,72,76].

Unfortunately, I didn't find any solution for *p*, *q*, *r*, and *s*, up to 1000. Maybe there is one for higher values, but that becomes too complicated.

With best wishes,

Comp.