MACRO OPERATORS

(*Comp meets Math in the cloakroom of the National Theatre after a performance of Nathan the Wise and tackles him*)

COMP. Hello Math, good to see you here! Better than in the Institute? Although I have good memories of our discussions about intuitive insights, perspicuous representations, and the like, I have long been going about with afterthoughts, and there is still one thing that intrigues me. We had many conversations but I don't remember that you ever mentioned macro operators. Yet I think that they are important enough for bringing them up in a discussion of problem-solving procedures in mathematics.

MATH. What a coincidence! Lessing already drew attention to macro operators when he discussed the use of verbs, and Jacob Israel de Haan, who knew this, once gave an example of a man visiting his doctor with a complaint about carrying out a painful, but seemingly unnecessary series of actions such as moving his right arm forward, moving his left arm backward and so on, whereas he could have described the situation as simply putting on his jacket. Yes, macro operators are important in problem solving, and I think that I must pay attention to them. Shall we continue the conversation in the theatre restaurant?

COMP. With pleasure.

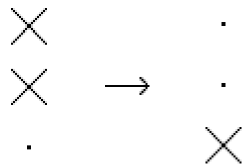
(*They go to the restaurant, take off their coats, and order a drink and a pastry*)

COMP. Do you know the paper by Iba about macro operators in peg solitaire? He presented it at the IJCAI conference of 1985.

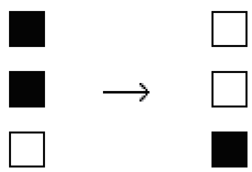
MATH. I am afraid I have never seen it. My association with macro operators is the eight-puzzle problem in which the end state can be reached by a certain rotation. (*He draws the following figure on a napkin*):

5	6	7
4		8
3	2	1

COMP. I know this example. It was given by my teacher in his course on the philosophy of AI. He doubted if a program based on SOAR would be able to generate the corresponding macro operator. Moreover he emphasized that human beings can describe this solution and communicate it to others in ordinary language, because of which he called it a common-sense solution. Notice that it takes thirty elementary moves! He also drew the attention to the paper of Iba, but he was also skeptical about Iba's claim about what his program had achieved. Anyhow Iba distinguished a small number of macros which I memorized, so that I could demonstrate my skill of solving the classical peg problem at birthday parties … It goes as follows. First of all we have the elementary moves, to begin with



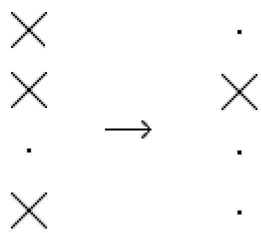
MATH. I see. It is not necessary to give the other ones, because they are reached by simple transformations. Is it useful to see these elementary moves as the creation of a negative image?



COMP. I don't know, but it is possible that a Boolean notation is interesting from a mathematical point of view:

$$110 \rightarrow 001$$

Moreover, what do you do with the following macro operator?



MATH. It amounts to a rotation over 180° and a negative image. You would describe it as

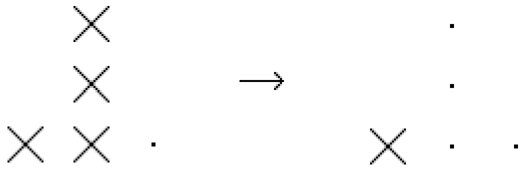
$$1101 \rightarrow 0100$$

But how did you remember this operation?

COMP. As ’dodd’, after John Dodd, a great English bow maker. As soon as I distinguish this configuration, I know that I can eliminate the end pieces.

MATH. Interesting. My first violin teacher played with a Dodd!

COMP. Lucky man! But now for the next macro:



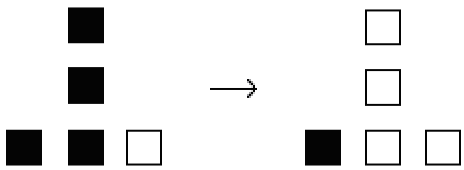
MATH. I see. It replaces three elementary moves. It can also be seen as a reflection in the central axis and a negative image, two operations for one macro. But your one-dimensional notation does not suffice any more.

COMP. Shall we from now on use a two-dimensional representation? Something like:

$$\begin{array}{ccc} x & 1 & x \\ x & 1 & x \\ 1 & 1 & 0 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} x & 0 & x \\ x & 0 & x \\ 1 & 0 & 0 \end{array}$$

The variable *x* can take any value, either zero or one. How would you describe this complex operation in ordinary language?

MATH. As soon as three pieces on a row are flanked at one end by two different elements, these three pieces can be deleted.



COMP. That is clear. But how would you describe the following very useful macro with which six pieces are taken away?

$$\begin{array}{ccc} 1 & 1 & x \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ x & x & 1 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ x & x & 1 \end{array}$$

MATH. Is this the only way of deleting a configuration of two by three pieces?

COMP. No, it is not. But I have no complete picture of the possibilities. I propose that we find out at home. Moreover it is already late. But I have a question for you. How did you know what Lessing said about verbs? Did you read his Hamburg Dramaturgy?

MATH. I once had a look into that work, but I have read about Lessing's remarks in the third volume of Mauthner's *Kritik der Sprache*. I will look it up. Shall we meet each other tomorrow at the Brouwer Institute?

COMP. Agreed!

(*They pay the bill and go their different ways*)

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(*Next morning in Comp's room*)

MATH. What did you find, Comp? And how?

COMP. When I came home yesterday night I tried to solve your problem, so I started with the following position:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ x & x & 0 \end{array}$$

and I discovered that it can be reduced to a position in which a rectangle of two by three pieces has been eliminated:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ x & x & 0 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ x & x & 0 \end{array}$$

also, that the piece above to the right had not been used:

$$\begin{array}{ccc} 1 & 1 & x \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ x & x & 0 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ x & x & 0 \end{array}$$

But then I did not want to go through with this, and it was also too late for making the computer do it, so this is all that I found.

MATH. You are right that it is not very interesting to proceed this way, but don't you think that your results can be generalized?

COMP: What do you mean?

MATH. Well, it might be that we can bring all possiblilities for the third column under one formula. I have the intuitive feeling that a rectangle of two by three can always be deleted as soon as it is accompanied by a four-place column containing the configuration

odd

What do you think?

COMP. That would be odd …

MATH. I have a proposal. You are going to use the computer in order to find out whether I am right or not, and I take pen and paper to do so.

COMP. OK. Here is a sheet of paper. Good luck!

(*After less than half an hour Math stops writing and sighs*)

MATH. Yes! I was right. How far did you come?

COMP. I am still working at it …

MATH. Do we now have enough macro operators to solve the traditional solitaire task?

COMP. We can try it out, look (*he has already set ready a 'real' game of solitaire*):

```
        1 1 1
        1 1 1
    1 1 1 1 1 1 1
    1 1 1 O 1 1 1
    1 1 1 1 1 1 1
        1 1 1
        1 1 1
```

MATH. It seems that there is as yet no useful possibility for an odd macro! Am I right?

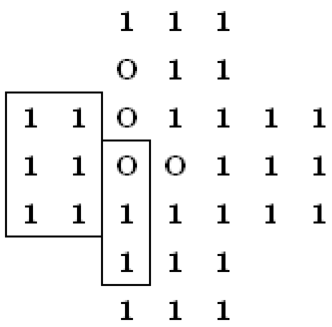
COMP. I always start with taking three pieces away:

```
        1 1 1
        1 1 1
    1 1 1 1 1 1 1
    1 1 1 O 1 1 1
    1 1 1 1 1 1 1
        1 1 1
        1 1 1
```

The result is clear:

```
        1 1 1
        O 1 1
    1 1 O 1 1 1 1
    1 1 O O 1 1 1
    1 1 1 1 1 1 1
        1 1 1
        1 1 1
```

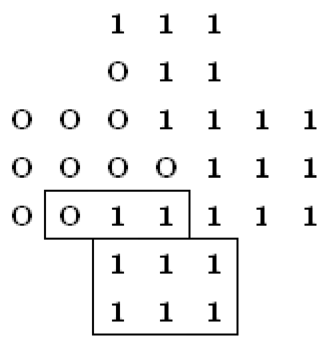
MATH. Now it is easy, or shall I say 'odd'?



I can take six away:

		1	1	1		
		0	1	1		
0	0	0	1	1	1	1
0	0	0	0	1	1	1
0	0	1	1	1	1	1
		1	1	1		
		1	1	1		

MATH. This is promising, for there is another odd macro:



Another six away:

		1	1	1		
		0	1	1		
0	0	0	1	1	1	1
0	0	0	0	1	1	1
0	0	1	1	1	1	1
		0	0	0		
		0	0	0		

COMP. Go on!

MATH. Odd again:

```
    1 1 1
    0 1 1
0 0 0 1 1 1 1
0 0 0 0 1 1 1
0 0 1 1 1 1 1
    0 0 0
    0 0 0
```

It becomes monotonous:

```
    1 1 1
    0 1 1
0 0 0 1 1 0 0
0 0 0 0 1 0 0
0 0 1 1 1 0 0
    0 0 0
    0 0 0
```

COMP. And now?

MATH. I take three away, as you did in the beginning:

```
    1 1 1
    0 1 1
0 0 0 1 1 0 0
0 0 0 0 1 0 0
0 0 1 1 1 0 0
    0 0 0
    0 0 0
```

and I am almost ready:

```
    1 1 1
    0 1 1
0 0 0 1 1 0 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

COMP. So far, so good, but I can tell you that the two remaining possibilities for taking three away are not feasible.

MATH. I see that too. Therefore I will continue with elementary moves, and see if there is still a solution.

COMP. How far can you look ahead?

MATH. Six moves must be possible. Let me see … (*Math pores over the board*) Yes!

```
    1 1 1
    0 1 1
0 0 0 0 0 1 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

COMP. Aha! Shall I …

MATH. Let me do it, I have come to like it!

```
    1 1 0
    0 1 0
0 0 0 0 1 1 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

Do you also see how it is finished?

```
    0 0 1
    0 1 0
0 0 0 0 1 1 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

COMP. Good old Dodd again!

MATH. I see it differently, but it is up to you!

COMP.

```
    0 0 0
    0 1 0
0 0 0 0 1 1 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

MATH. At last we have the tiny macro we began with yesterday night:

```
    0 0 0
    0 0 0
0 0 0 0 0 0 0
0 0 0 1 0 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

COMP. Fine!

MATH. An interesting aspect of macros is that they may make a particular solution perspicuous. This was the case with the eight-puzzle problem which I mentioned yesterday. However our solution of the solitaire problem fails in this respect. I had to insert an elementary move after the second application of the elimination of three in a row.

COMP. Iba had a macro for that situation. He called it ‘Remove-L’:

```
    1 1 1
    0 1 1
0 0 0 1 1 0 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

MATH. That was clearly ad hoc, and I understand why your teacher had misgivings about Iba’s achievements. And why was Iba satisfied with this solution? A mathematician would have looked for a truly perspicuous solution, and if he could not find one, then he would have told so.

COMP. It is possible that Iba tried to find out whether there is a solution with a minimum of different macros. If that is indeed the case, he has probably got the result that his ‘Remove-L’ macro was always needed at the end of the game.

MATH. The description ‘Remove-L’ does not mention the precondition of the operator. I see no simple remedy for this, so I remain critical about such a macro, not to mention that it is used only once.

COMP. Does this mean that the whole project has failed?

MATH. Not at all. I noticed that so far all macros can be described as operations with which only pieces are taken away and none are added. This looks plausible, but it has led to an obscure macro, the ‘Remove-L’. Moreover the corresponding solution could not do without an elementary move, although only at the end.

COMP. When you looked six moves ahead, you imagined only elementary moves one after another, whereas I noticed a dodd situation after your first move (*Comp reconstructs the state which preceded it*):

```
    1 1 1
    0 1 1
0 0 0 1 1 0 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

It is possible to regard the elementary move as a preparatory move for a dodd situation:

```
    1 1 1
    0 1 1
0 0 0 0 0 1 0
0 0 0 0 1 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

(*Comp applies the dodd macro*)

```
    1 1 0
    0 1 1
0 0 0 0 0 1 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

MATH. Then I would jump over two pieces, as if we were playing draughts:

```
    0 0 0
    0 1 0
0 0 0 0 1 1 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
    0 0 0
    0 0 0
```

The fact that I could look ahead can perhaps be explained by my experience with draughts. I did not tell you, because I would not lead you away from your macros.

COMP. There is indeed a difference between human players and computers. You could bring the game to a good end in your way. On the other hand a suitable computer program would not have problems with checking the precondition of the Remove-L macro and then applying it.

MATH. Would the draughts approach not lead to significant macros? To begin with:

```
1 1 0         0 0 0
    1    ⟶        0
    0             1
```

COMP. Why not? Look:

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ & & 0 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ & & 1 \end{array}$$

MATH. What do you think of this:

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & 0 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{array}$$

COMP. I see in which direction you want to go. Iba's macros are perhaps significant for a particular solitaire problem, but it is not clear whether they are interesting from a mathematical point of view.

MATH. Macros can indeed be judged from different points of view, mathematical, computational, and also psychological. When we restrict ourselves to mathematics, we would expect that macros have certain mathematical properties, depending on the task which they are supposed to perform. Macros in solitaire games can be judged by the criterion how far they contribute to solutions of solitaire puzzles. I would define such a puzzle as the problem of transforming a finite subset of the two-dimensional raster with exactly one empty place, into itself with exactly one occupied space, with the usual transformation rule. Classical solitaire can be defined as a solitaire puzzle with the property that the goal state is the negative of the initial state. But then we can also omit the condition that the initial state has only one empty space. This would provide more solvable solitaire problems.

COMP. Can you give an example?

MATH: A very simple one is the following problem:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

COMP. This produces another macro! And apparently the description you gave of the elementary move as the forming of a negative image was not at all ridiculous. Another question: do you think that initial states in the form of a square or rectangle are interesting?

MATH. We can start with the situation of a moment ago. The third row can be reached by your famous dodd macro, so let us add one column and apply it:

$$\begin{matrix}1&1&1&1\\1&1&1&1\\1&0&1&1\end{matrix}\quad\longrightarrow\quad\begin{matrix}1&1&1&1\\1&1&1&1\\0&0&1&0\end{matrix}$$

It follows that the whole operation comes down to a solution of a classical solitaire puzzle.

$$\begin{matrix}1&1&1&1\\1&1&1&1\\1&0&1&1\end{matrix}\quad\longrightarrow\quad\begin{matrix}1&1&1&1\\1&1&1&1\\0&0&1&0\end{matrix}\quad\longrightarrow\quad\begin{matrix}1&0&0&0\\1&0&0&0\\0&1&0&1\end{matrix}\quad\longrightarrow\quad\begin{matrix}0&0&0&0\\0&0&0&0\\0&1&0&0\end{matrix}$$

COMP. Interesting! You started with two macros and combined them in such a way that a solvable solitaire puzzle came about.

MATH. We can even go further by adding two rows to the second last state, and then applying a odd macro:

$$\begin{matrix}1&0&0&0\\1&0&0&0\\0&1&0&1\\1&1&1&1\\1&1&1&1\end{matrix}\quad\longrightarrow\quad\begin{matrix}0&0&0&0\\0&0&0&0\\1&1&0&1\\1&1&1&1\\1&1&1&1\end{matrix}\quad\longrightarrow\quad\begin{matrix}0&0&0&0\\0&0&0&0\\1&1&0&1\\1&0&0&0\\1&0&0&0\end{matrix}$$

COMP. It is a pity that the total result is not a classical solitaire puzzle:

$$\begin{matrix}1&1&1&1\\1&1&1&1\\1&0&1&1\\1&1&1&1\\1&1&1&1\end{matrix}\quad\longrightarrow\quad\begin{matrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{matrix}$$

But the procedure opens perspectives. Instead of making an attempt to solve a particular problem with the risk that there is no solution at all, solvable problems are constructed from solved problems. It reminds me of a similar approach which my teacher in the philosophy of AI chose for finding solvable rectangular boards for the problem of the knight, which is how to reach every cell in a tour of the knight around the board.

MATH. I know. This problem also has a classical form in the requirement that the knight should be brought back to his initial position, in a so-called closed tour. But already the board of 4 by 3 is not classical, although one of its solutions can be used to solve the problem for a board of 8 by 3. A similar success can be reached by our last solution. Do you see it?

COMP. Of course. I will add a rectangle of 4 by 5 and repeat the last solution:

$$\begin{array}{cccccccc} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{array} \longrightarrow \begin{array}{cccccccc} 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&1&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1 \end{array}$$

I apply the dodd macro to the result and then it is obvious what to do:

$$\begin{array}{cccccccc} 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1 \end{array} \longrightarrow \begin{array}{cccccccc} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}$$

MATH. Now we know that this is a solvable problem we can try to solve it with other macros such as dodd and odd.

COMP. (*After some scribbling*) No problem. Only at the end I had to look for suitable elementary moves.

MATH. I think that a computer program should also start with such macros, but must solve the endgame without them. However, how does it decide that the endgame has arrived?

COMP. When the macros do not work anymore. The last problem seems a good example for trying this out. When I solved it in my way, the endgame arose when there were only five pieces left.[1] How would this be with the computer?

MATH. I would suggest that you try to answer this question, whereas I am going to think about the use of macros in theorem proving.

COMP: Good idea! Just one question: did you find the reference to Lessing?

MATH. First of all I looked into the doctoral thesis of Jacob Israel de Haan. His quotations from Lessing were taken from Mauthner, so I consulted the third volume of Mauthner's *Beiträge zu einer Kritik der Sprache*. It appeared that Lessing formulated his definition of an action in his essays on the fable, and he gave a philosophical comment in the sixteenth chapter of *Laocoon*. I wrote the definition out,[2] and will now try to give a translation:

> I call an action a succession of changes which together constitute a whole. This unity of the whole rests on the conformity of all parts to a final goal.

COMP. Nice quotation! Perhaps we can use it for a definition of a macro operator. I will think about it.

MATH. So will I. See you later. (*He leaves Comp's room*)

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[1] This was Comp's endgame:

0	0	1	0
0	1	1	0
0	1	1	0

[2] Eine Handlung nenne ich eine Folge von Veränderungen, die zusammen ein Ganzes ausmachen. Diese Einheit des Ganzes beruhet auf der Übereinstimmung aller Teile zu einem Endzweck.