INTRIGUING INFERENCES

(*Math, Log and Comp decided to have regular discussions with each other. This time they are together in Math's room. Apparently Log wants to raise questions about types of inferences, a hitherto neglected subject.*)

LOG. Your favourite subjet is intuitive inferences, Math, and last time we talked about part-whole inferences, but I haven't heard much about logical inferences. Do you consider them less important?

MATH. Not at all, Log. Yet they form only a subclass of inferences in mathematics.

LOG. What do you mean? I thought that mathematical proofs consist solely of logical deductions from axioms and theorems.

MATH. That is not my point. I am interested in problem solving procedures and that is a quite different matter. For me anything counts that contributes to a solution of a mathematical problem. So I distinguish not only deductive inferences, but also inductive, abductive and analogical inferences, because all these types of inference occur in mathematical problem solving. Mathematical problems, by the way, are not restricted to deductive problems, although adepts of Artificial Intelligence sometimes act as if all problems are state-space problems.

COMP. I know, but I have an excellent *Introduction to Artificial Intelligence* by Philip Jackson, who introduced the name 'system inference problems' for a type of problems that can certainly not be solved by logical reasoning alone. I understand that your other types of inference are used in these kinds of problems, but what is the connection?

MATH. Not too fast. The distinction between state-space and system inference problems is not exhaustive. There are also proof finding problems, problem making problems, and theory construction problems. And this makes the distinction between the types of inference that I mentioned only more significant.

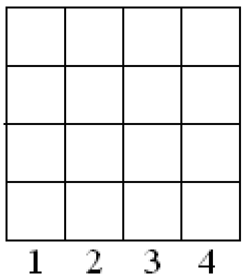
LOG. I would like to see examples.

COMP. Please keep it simple, Math. I remember that I had great difficulties to follow your juggling with polygonal numbers.

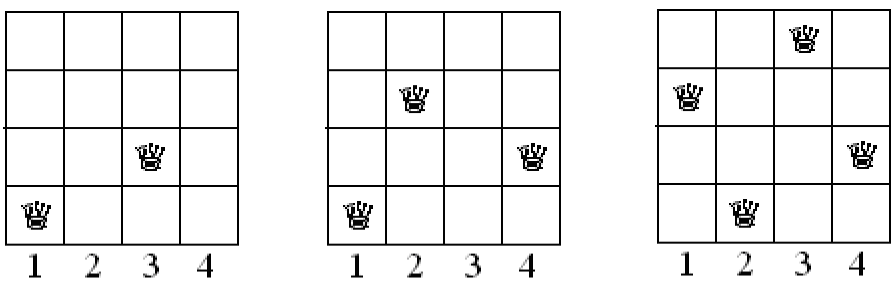
MATH. All right. I will give an example of an extremely simple state-space problem that will soon convince you that other than deductive inferences are needed. It is the old problem of the queens, or the problem of placing *n* queens on a board of *n* by *n* squares so that no one of them can take any other in a single move.

COMP. When I was a student, I met the eight queens problem in the introductory course in programming. The last chapter of our course book contained a page with all solutions for this special case of the *n* queens problem. Do you have a reason for posing the general problem? It seems to me that the eight queens problem is exemplary for the rest.

MATH. You may be right from the point of view of a computer scientist. But as a mathematician I want to solve all *n* queens problems at once, that is to say, I want a general solution. Therefore I do not even start with the eight queens problem, but with the four queens problem, however childish this may appear to you. I start with an empty board. Look! (*Math goes to his new whiteboard*)

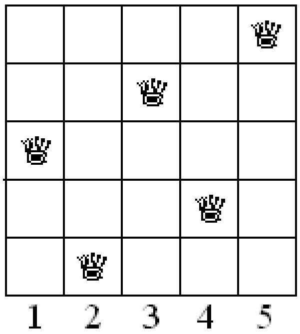


I am going to place the queens in the rows, beginning with the lowest one. Now, if the place of the first queen is in the first column, then the place of the second queen is either in the third column or in the fourth column. But if we put the second queen in the third column, then there is already no place left for the third queen. You see, this is a question of deductive inferences. If you don't mind I will draw the different end positions on the whiteboard:



LOG. I am deeply impressed, Math. What are you up to?

MATH. First of all, I hope that you see that we can easily derive a solution for the five queens problem from the solution of the four queens problem. We need only to enlarge the board and put the fifth queen in the new corner:



Secondly, I want to draw your attention to the fact that I gave only two possibilities to the first queen. We know that the other two amount to the same, because of the symmetry, but I am afraid that a computer would not discover this by itself.

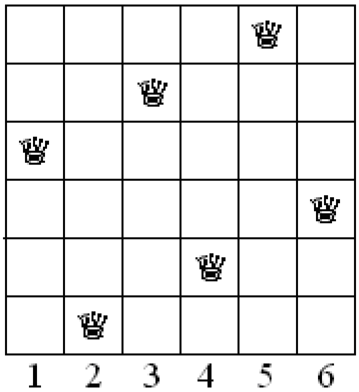
COMP. What you mean by this is that a computer program which has only the possiblity of, say, breadth first search, will start with four possibilities for the first queen. But it is possible to program it in such a way that it does not take into account the symmetrical positions.

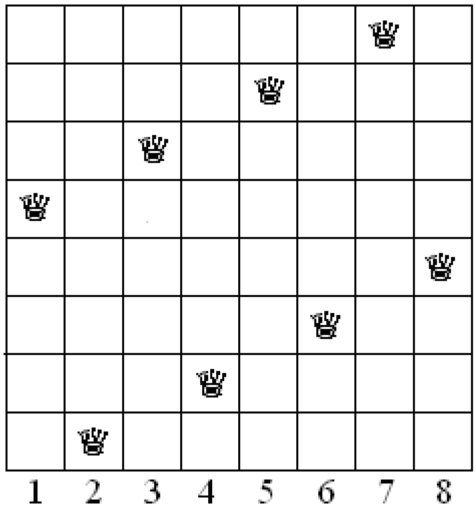
MATH. But we human beings see this immediately and need no proof that each solution found in this way has a symmetrical counterpart. But this aside.

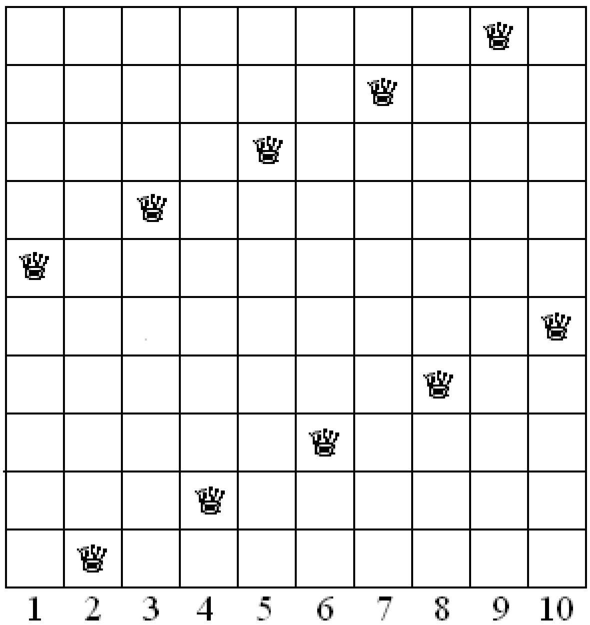
Thirdly, I hope that you saw that the solution is also symmetrical: the upper half can be derived from the lower by a rotation. This suggests the idea that we should limit the solutions to regular ones, for example such as can be obtained by successive knight's moves in the lower part of the board. Now the first queen of our regular solution of the four queens problem stands in the second column. Can you form hypotheses about the corresponding place for the general problem?

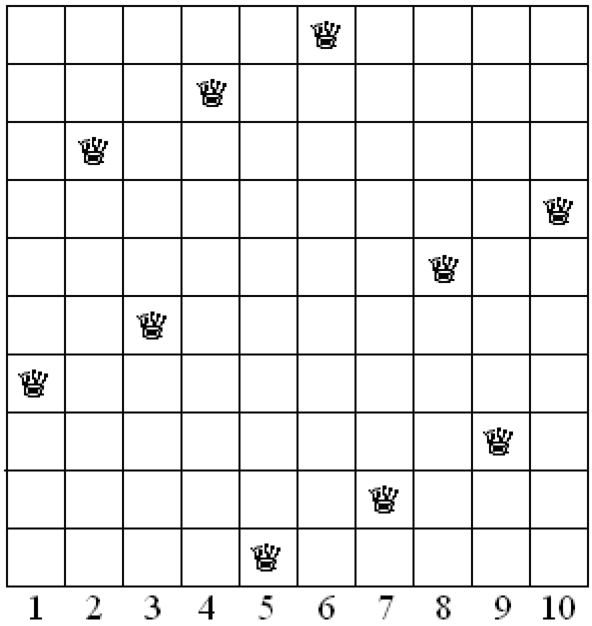
LOG. Let me introduce a simple notation, and indicate the place given to the kth queen by p(k). Then it is obvious that there are two simple hypotheses, p(1) = 2, and p(1) = ½n. Is that what you mean?

MATH. Yes, and my congratulations, for you have just drawn two inductive inferences. Let us try them one by one, restricting ourselves to even values of n, expecting that solutions for odd values of n can derived in the indicated way. Well, I give you the results of the application of the hypotheses for the values 6, 8, and 10. (*Math draws the following four figures.*)









MATH. You see that the first hypothesis holds for $n = 6$, the second for $n = 8$, whereas both hypotheses hold for $n = 10$. What do you expect for $n = 12$?

COMP. The first hypothesis?

MATH. Yes indeed. And for $n = 14$?

COMP. The second hypothesis. And now you are certainly going to argue that both hypotheses hold for $n = 16$.

MATH. It is even worse. The first hypothesis holds for $n = 6$, 12, 18, and so on, that is for all values of the form $6m$, the second hypothesis for $n = 8$, 14, 20, and so on, so for all values of the form $6m + 2$, and both hypotheses for all values of the form $6m – 2$. If you do not believe me, you can try these solutions yourself for as many values as you want. But notice that the conclusions are due to an inductive inference. Isn't that a confirmation of my claim about the significance of other types of inference than deductive ones?

LOG. I assume that you already took the pains to prove that the inductive conclusions are correct.

MATH. Yes, I did prove it, but my proofs are too long to reproduce them here. I should like to confine myself to the remark that they required rather complicated algebraic operations and some elementary number theory. What is more important: the proofs did not come of themselves.

That is characteristic of abductive inferences, as those inferences are called in which suitable premises are framed for given conclusions. I wonder whether students of Artificial Intelligence can offer alternative procedures that computers can perform.

COMP. There exist automated induction performances, but I doubt if abduction can already be automated.

LOG. Do you think that a computer program can get away with such formulas as – let me write them down (*she goes to the whiteboard*):

$f(4) = (2, 4, 1, 3)$
$f(6) = (2, 4, 6, 1, 3, 5)$
$f(8) = (4, 6, 8, 2, 7, 1, 3, 5)$
$f(10) = (2, 4, 6, 8, 10, 1, 3, 5, 7, 9)$ or
$f(10) = (5, 7, 9, 1, 3, 8, 10, 2, 4, 6)$

MATH. We mathematicians need only a few pictures in order to make good guesses. That does not cease to be amazing. How they sometimes succeed to solve proof finding problems is another question. In my case, I started by formulating the general solutions for $n = 6m$ and so on, and finally I succeeded to get the proofs done, but it is too long ago for me to remember how I found them.

LOG. I can understand. But now I am anxious to hear what you have to tell about problem making problems.

MATH. My present example is the result of an analogical inference. You see that the knight's move is characteristic of the given regular solutions of the n queens problem. But can we also solve it without knight's moves, that is, in such a way that none of the queens is at a knight's move distance from any other queen? This is my new problem, but I admit that I have not yet worked on it.

LOG. How do you know that it has a solution at all?

MATH. Well, we can try to find a solution in a particular case, say $n = 10$.

COMP. I can do the same, although I will consult my computer, so to say. After all, it will have no problem with finding a particular solution of such a simple state-space problem.

LOG. Let us see who comes first with a solution, assuming that there is one. I hope, of course, that there is no solution at all. Then we would have an interesting proof finding problem!

MATH. Nevertheless I will try. (*He takes pen and paper.*)

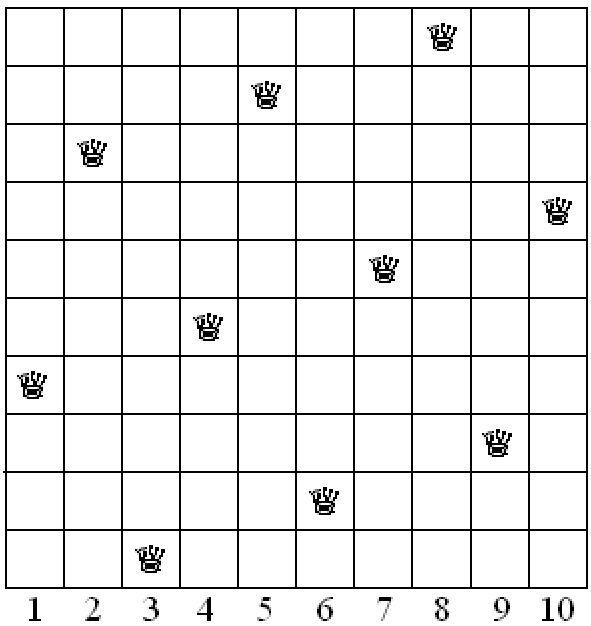
COMP. So shall I. (*He leaves the room.*)

LOG. I will look into some books of your library, Math, if you don't mind.

(*Already Math does not hear what Log is saying, so Log takes a book with a red cover from the shelf. After a while Math cries 'Yes' and at the same time Comp appears in the doorway. He is laughing and notices that Log is smiling when she looks away from her book.*)

LOG. I understand that both of you found a solution. So did I, but I found it in this book. Please show me your solution, Comp, and I ask you, Math, to draw your solution on the whiteboard. (*They do what she asks.*)

MATH. Here it is:



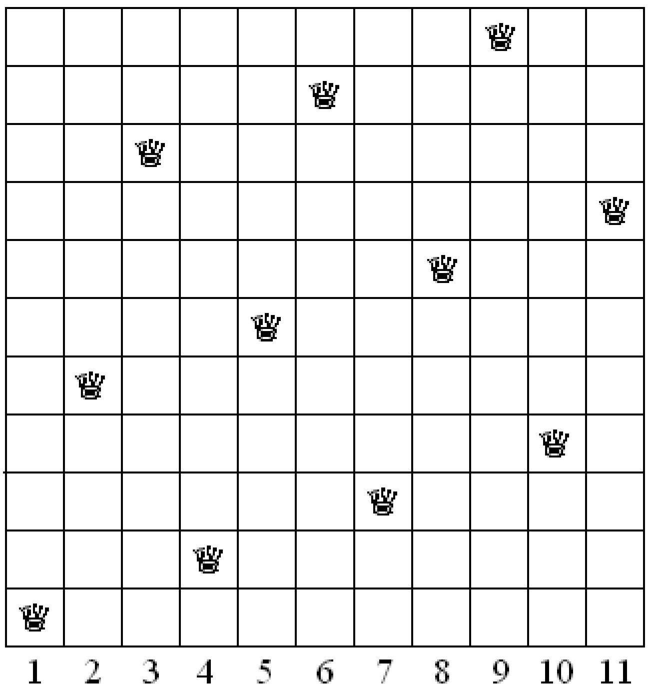
LOG. That is amazing! This is Comp's solution:

3, 6, 9, 1, 4, 7, 10, 2, 5, 8

MATH. Shall I tell you how I found my solution? It spontaneously occurred to me that I could possibly find a regular solution by taking a prolonged knight's move. I experimented first with an eight by eight board, but that did not work, so I turned to the ten queens problem. It was obvious that it made no sense to put the first queen in the left corner, for that would not give a regular solution with rotational symmetry. Therefore I tried p(1) = 3, deliberately skipping the second column, because I had the feeling that p(1) = 2 was reserved for regular solutions with the normal knight's move. I even had an association of the number three and the prolonged knight's move, which has also a three in it. To my astonishment I got the solution right on.

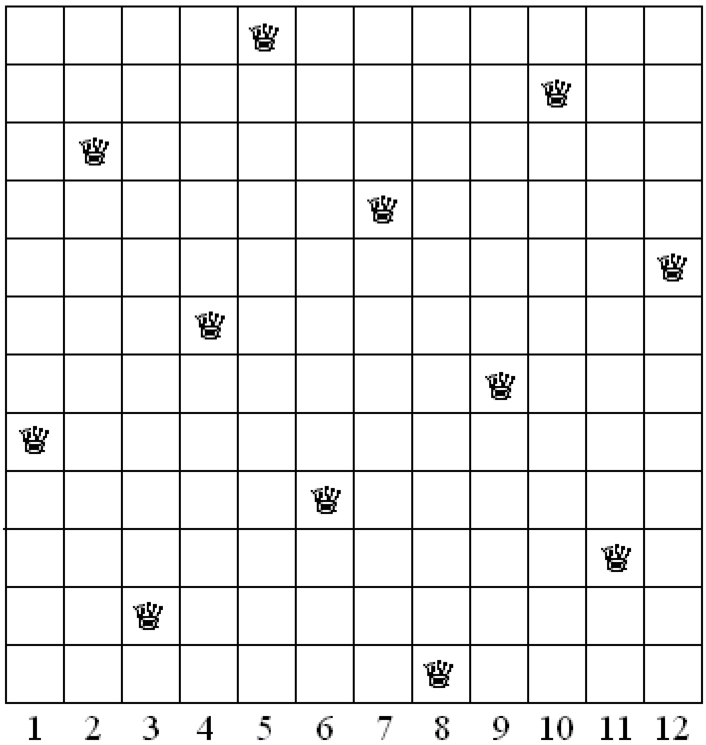
COMP. I found the solution after I had checked that the solutions of the eight queens problem in the introductory course book that I mentioned before did not contain a solution without a knight's move. Then I wrote a simple program for the ten queens problem and Log reproduced the first solution that it gave.

LOG. It seems to me that Comp's approach is not very different from Math's. Only Math used his famous intuition, so to say. But there is an element in his explanation that may be worthwhile to pursue further, especially when it comes to proving a general rule. Math said that he deliberately put the first queen in the third column. It looks as if this is the place of the second queen in an eleven queens problem. Look (*she adds a column and a row to the previous diagram*):

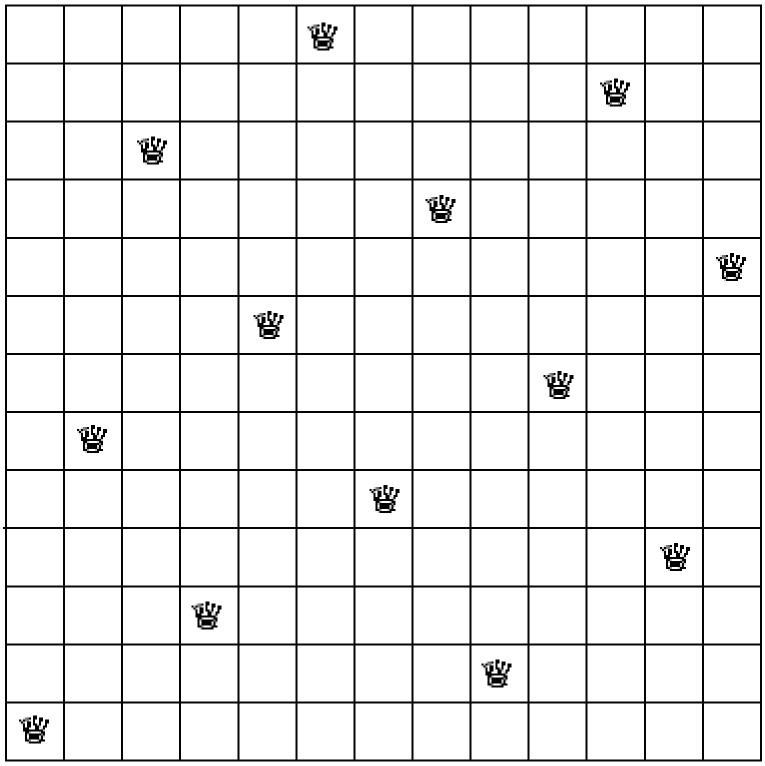


MATH. Wow! That looks promising!

LOG. I did not tell you that I also found a solution of the 12 queens problem in the book that I took, *Mathematical Recreations* by Maurice Kraitchik. Here it is (*she copies a part of Figure 126 of the book*):



COMP. It is obvious that this solution can be derived from a solution of the thirteen queens problem in the manner of what Log saw a moment ago:



LOG. This solution begins in the left corner below and repeatedly uses a translation 8 horizontal, 1 vertical. I would call it a periodical solution. It is not symmetrical, unlike the derived solution of the twelve queens problem

MATH. I did not yet mention it, but I regard concept formation too as a type of inference that may be useful for solving theory construction problems. When we look at the regular solutions of the even queens problems, we see that they are all symmetrical, but only some of them are periodical. Those that are not can be derived from periodical solutions of odd queens problems. I think that this might of great help, if we want a theory for regular solutions in general. I would suggest that we focus on periodical solutions, because they have an obvious connection with number theory. If that is correct, then we have to reverse our initial derivation of a solution of the five queens problem from a solution of the four queens problem. The former was periodical, but the latter was not.

COMP. Do you mean that we must start all over again? I am sorry, mathematical recreations are nice, but I have more to do.

LOG. So have I. See you, Math.

(*They leave Math behind in puzzlement*)