FINITE GEOMETRIES

The method of transposition, that is, solving problems after transposing them to another domain, has proved to be useful in both finite projective and finite affine geometry.[1] These are 'pure' line geometries: points are seen as parts of lines, and there are no other geometrical figures than points and lines. An overview will be given in section one. But we can also define 'circle geometries' by postulating that points are to be seen as parts of circles and then the obvious question is whether the method of transposition can also be fruitfully used in this field of research. This is the subject of section two. In section three the question is addressed whether the results can be generalized for 'curve geometries' in general.

Finite line geometries

The main idea of 'transpositional tricks' in line geometries can easily be demonstrated by the problem of finding models for axioms such as

1. for each two distinct points, there is exactly one line containing them both
2. for each two distinct lines, there is exactly one point contained by both
3. not all points are on the same line
4. there exists at least one line
5. every line contains exactly 3 points[2]

For someone who is not familiar with this kind of problem: the 'classical' model is Fano's famous 'projective plane' (Figure 1):

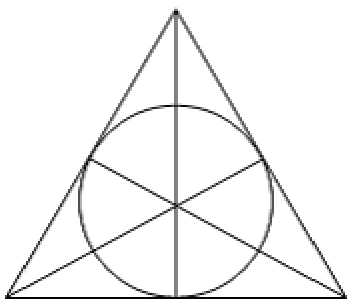

Figure 1

Presumably it was found by Fano step by step, and by trial and error, starting with three points on a line.

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[1] See Henk Visser (2001), Transpositional tricks, BNVKI Newsletter, Vol. 18, No. 2, reprinted in the ALP Newsletter (14)3.

[2] There is no need for transpositions if it is required that every line contains exactly two points: just imagine a triangle.

But the problem can also be systematically solved by representing seven points on a circle and then choosing a triangle, representing a line containing three points, in such a way that it has exactly one point in common with each of its rotations over $2k\pi/7$ ($k = 1, 2, 3, \ldots, 6$) around the centre of the circle (Figure 2).

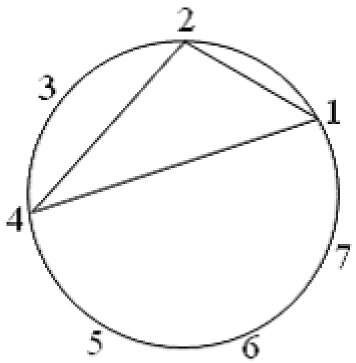

Figure 2

This problem may in its turn be ‘transposed’ to the problem of writing 7 as a sum of three numbers in such a way that each number below 7 appears exactly once as a partial sum, including cyclical combinations, in order to achieve that there are no more overlappings than in one single point:

$$7 = 1 + 2 + 4$$

$1 = 1$
$2 = 2$
$3 = 1 + 2$
$4 = 4$
$5 = 4 + 1$ (cyclical)
$6 = 2 + 4$

The resulting model can be pictured in different ways. First of all, Figure 2 can be ‘completed? by adding the other six triangles, preferably in different colors.

There is also the following ‘numerical’ representation:

124
235
346
457
561
672
713

The cyclical character of the model is conspicuous. Notice that it has been lost in Fano's representation. Therefore I prefer the following representation in which each point possesses two 'locations' (Figure 3):

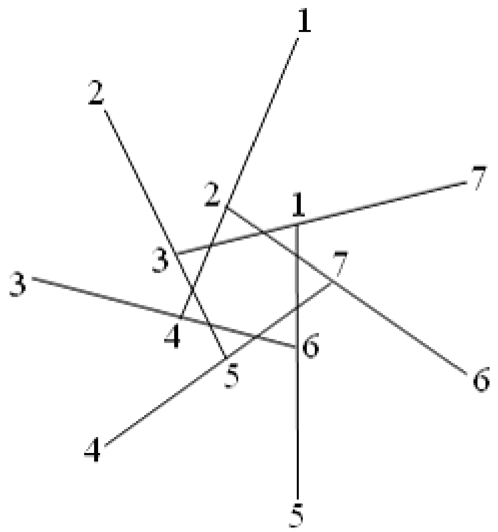

Figure 3

By varying axiom 5, for example postulating that every line contains exactly 4 points, we get models for the resulting geometry by the following partitions of 13:

$$13 = 1 + 2 + 6 + 4$$
$$13 = 1 + 3 + 2 + 7$$

There is no need to pursue the matter further. In this way the so-called finite *projective* 'line geometries' may acquire models, though it can be proved that not every variation of axiom 5 leads to a solution. For example, there is no projective line geometry in which every line contains exactly 7 points.[3] The corresponding number of points, to wit 43, has no partitions with the required property.

Similar considerations hold for so-called finite *affine* line geometries. Here the above axiom 2 is replaced by the following axiom:

2*. Through a point not on a given line there is exactly one line which does not meet the given line

It is easy to make a model for such a geometry if it is postulated that every line contains exactly 2 points: just imagine a tetraeder. But the

[3]This result is due to my colleague Dr. Donkers, whose java-program solved the partition problem for projective line geometries up to 10 points on every line.

problem of finding a model when every line contains exactly 3 points is difficult. Therefore it seems practical to shift to a domain of numbers and to solve the corresponding combinatorial problem, possibly with the following result:[4]

123 246 349 478 569
145 258 357
167 279 368
189

However, there is little 'structure' in the thus acquired solution, so let us turn to a representation with a circle, this time with center 1 and eight points 1, 2, ..., 8 on it. Now it is possible to draw three 'lines', that is, triples of points, that do not meet each other (Figure 4):

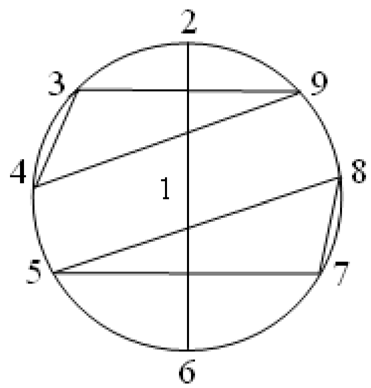

Figure 4

Starting with the line 126, we see that there are six points that are not 'on' it, but through each of these points there is exactly one line which does not meet the given line. This has been done in such a way that only two more lines, 349 and 578, are needed. Now we rotate the three lines over $\pi/4$ and we get the lines 137, 452 and 689. In the same way, we get the lines 148, 563, 792 and 159, 674, 823. In sum:

126 349 578
137 452 689
148 563 792
159 674 823

It is important that for each two distinct points, there is exactly one line containing them both. This is reflected in the property of the partition

[4] John Wesley Young, *Lectures on Fundamental Concepts of Algebra and Geometry*. New York: The Macmillan Company, 1923, p. 42.

$$8 = 1 + 5 + 2$$

that each number under 8 except 4 appears exactly once as a partial sum, including cyclical combinations:

$$1 = 1$$
$$2 = 2$$
$$3 = 2 + 1$$
$$5 = 5$$
$$6 = 1 + 5$$
$$7 = 5 + 2$$

There is also a perspicuous representation in which each point has two locations (Figure 5):

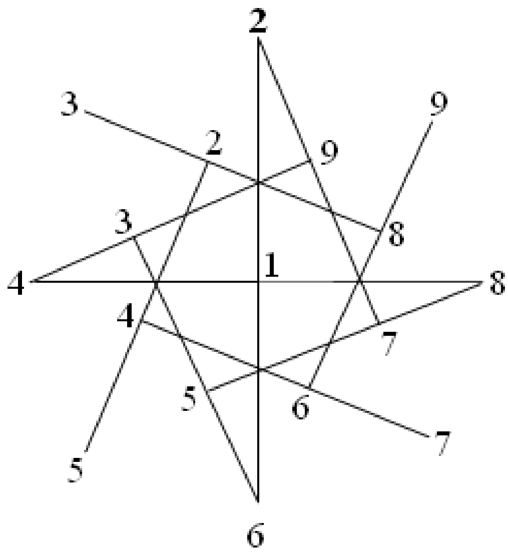

Figure 5

For ‘higher order’ finite affine line geometries it is useful to look for partitions of the number that precedes the number of points. For example, if every line should contain exactly 4 points, there is the following nice partition of 15:

$$15 = 1 + 2 + 4 + 8$$

Notice that there are no partial sums for 5 and 10. As a result, the corresponding geometrical picture with two ‘generating’ lines is as follows (Figure 6):

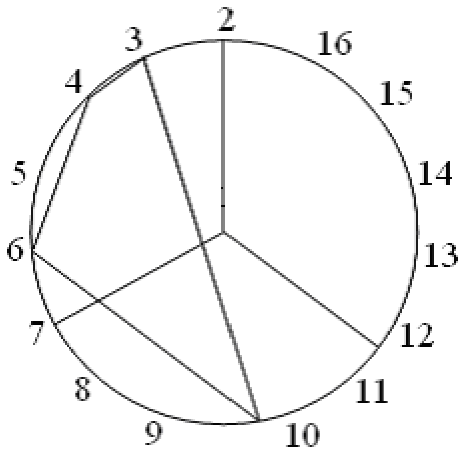

Figure 6

If every line should contain exactly 5 points, there are two partitions of 24:

$$24 = 1 + 2 + 8 + 9 + 4$$
$$24 = 1 + 3 + 5 + 2 + 13$$

However, there is no model of the affine line geometry in which every line contains exactly 6 points.[5]

The fact that some finite projective or affine geometries 'do not exist' is no reason to regard the whole system as insignificant, on the contrary: there are 'enough' models and it is a challenge to delineate and explain the exceptions. But even when there would have been 'few' models, their explanation might have been a serious question. Let us now see if this occurs with finite circle geometries.

Finite circle geometries

In order to get an idea of some possibilities of projective circle geometry, it is wise to consider the following axioms:

I. for each two distinct points, there are exactly two circles containing them both
II. for each two distinct circles, there are exactly two points contained by both

That it is simple to satisfy these first two axioms, is shown by the following picture (Figure 7):

[5] It is again Dr. Donkers who also wrote a program for the partition problem in finite affine line geometries.

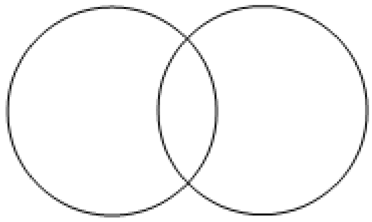

Figure 7

As usual, trivial models are eliminated by two more axioms:

III. not all points are on the same circle
IV. there exists at least one circle

If we want as few circles as possible, we get the following model (Figure 8):

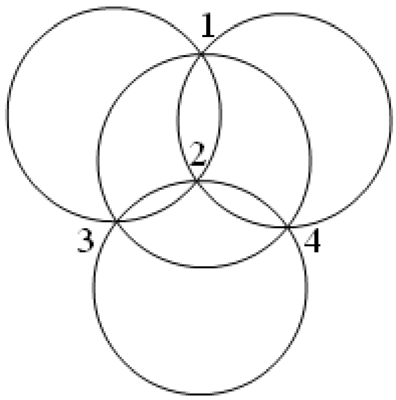

Figure 8

It also satisfies a special axiom:

V. every circle contains exactly three points

Moreover:

VI. for each three distinct points, there is exactly one circle containing them all
VII. for each three distinct circles, there is exactly one point contained by all

Notice that there are only four points and four circles. But as soon as there are more points, the first two axioms (I, II) and the last two axioms (VI, VII) cannot be satisfied simultaneously. (Suppose that there are five points, 1, 2, 3, 4 and 5, then there are not only circles through 1, 2 and 3 and through 1, 2 and 4, but also through 1, 2, and 5 according to axiom VI, but this would imply that there are three circles through the points 1 and 2, contrary to axiom I.) Moreover, axiom VI is not a familiar one, as my colleague Floris Wiesman remarked. What he had in mind was the Euclidean proposition that for each three distinct points *not on one and the same line* there is exactly one circle containing them all.

Before going to projective circle geometries with more than three points on every circle, we can have a look at the following obvious numerical representation of the model of Figure 8:

123
234
341
412

Its cyclical nature suggests the following representation of the circle formed by the first three points – in the form of a triangle that can be rotated to get the representations of the other circles (Figure 9):

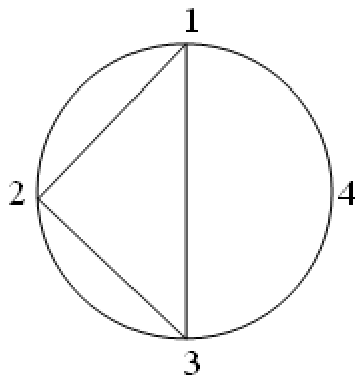

Figure 9

Apparently it can be regarded as the result of a partition, namely that of 4:

$$4 = 1 + 1 + 2$$

It is easily verified that each number below 4 appears exactly twice as a partial sum, and this is just as it needs to be, if we want to satisfy axiom 2. This suggests in its turn that models for line geometries with more than three points on every circle may be found by partitions with this property.

Suppose we require that every circle contains exactly four instead of three points:

V(4). Every circle contains exactly four points

In order to construe a model for the axioms I, II, II, IV and V(4), we look for a suitable partition of 7. (7 is one more than the number of circles that each have two points in common with the circle through the points 1, 2, 3 and 4.) We find:

$$7 = 1 + 1 + 2 + 3$$

It follows that the seven circles can be read off from the following picture (Figure 10):

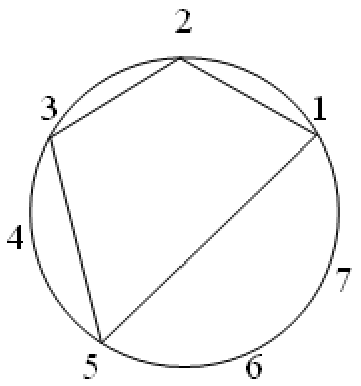

Figure 10

This picture is remarkable, because the points that are as yet unconnected, 4, 6, and 7, form a triangle that generates all the lines of a model for the projective line geometry we started with (Figure 11):

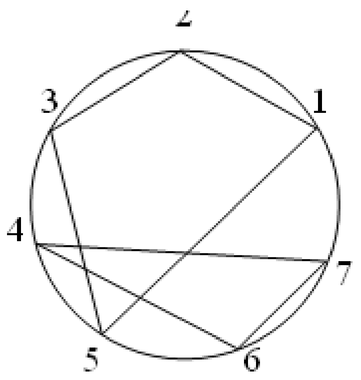

Figure 11

Therefore our model for the projective *circle* geometry with the property that every circle contains exactly four points and the model for the projective *line* geometry with the property that every line contains exactly three points can be combined in a model for the following axiom system:

1. for each two distinct points, there is exactly one line containing them both
2. or each two distinct lines, there is exactly one point contained by both
3. not all points are on the same line
4. there exists at least one line
5. every line contains exactly three points
6. for each two distinct points, there are exactly two circles containing them both
7. for each two distinct circles, there are exactly two points contained by both
8. not all points are on the same circle
9. there exists at least one circle
10. every circle contains exactly four points
11. for each three distinct points not on the same line there is exactly one circle containing them all

Now we understand that Fano's projective plane can be extended to the following perspicuous picture (Figure 12):

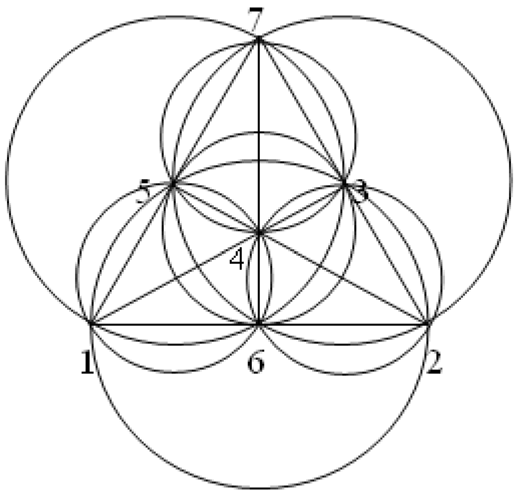

Figure 12

(Notice that the points 3, 5, 6 form a "line", and the points 7, 1, 2, 4 a 'circle'.)

Together there are seven circles and seven lines, which we can represent numerically as follows:

1235 467
2346 571
3457 612
4561 723

5672 134
6713 245
7124 356

Again we can modify the axiom that every circle contains exactly four points into

V(5). Every circle contains exactly five points

In order to construe a model for the axioms I, II, II, IV and V(5), we look again for a suitable partition, this time a partition of 11, and the first guess is the hit on the nail:

$$11 = 1 + 1 + 2 + 3 + 4$$

The eleven circles can be extracted from the following picture (Figure 13):

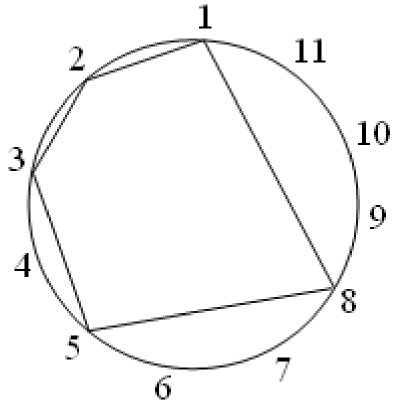

Figure 13

There are no theoretically different partitions than the given one. It was again my colleague Jeroen Donkers whose computer program also checked the possibilities for partitions in order to find models for the projective circle geometries with, respectively, six, seven, eight and nine points on every circle. Dr. Donkers found that only the last geometry, that is the set of axioms, I, II, III, IV and V(9), has models:

$$37 = 1 + 3 + 2 + 4 + 5 + 2 + 1 + 7 + 12$$
$$37 = 1 + 2 + 4 + 10 + 7 + 1 + 4 + 6 + 2$$

We leave projective circle geometry and proceed to projective affine geometry. The central idea is, of course that every circle has exactly one ‘opposite’, that is a circle which has no points in common with it. So if there is a circle through three given points, then there are three more points that are contained by its opposite, together already six points. This leads to the question whether we can make minimal models with exactly six points. Fortunately we have learned from affine line geometry that we have to draw a circle with five points on it, because its centre also represents a point of an affine model (Figure 14):

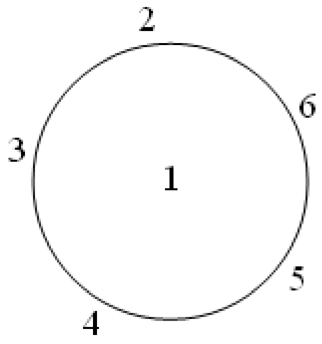

Figure 14

There are exactly two partitions of 5:

$$5 = 1 + 1 + 3$$
$$5 = 1 + 2 + 2$$

Both lead to a set of circles, generated by the following pictures (Figure 15 and 16):

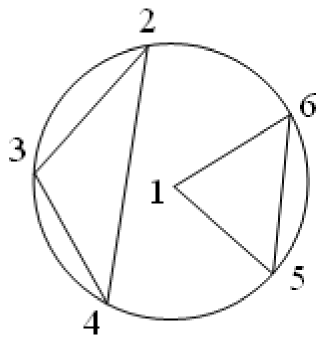

Figure 15

234	156
345	162
456	123
562	134
623	145

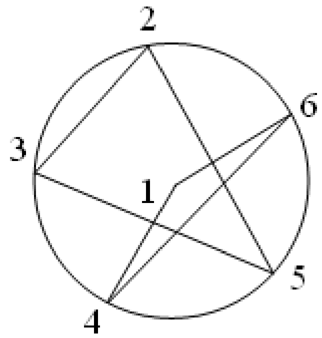

Figure 16

235	146
346	152
452	163
563	124
624	135

I did not call these sets 'models', because we have as yet not made our choice for an axiom sytem. This is an advantage if we want an affine circle geometry that is as 'promising' as possible, that is, still has models for axioms of the form 'every circle contains exactly *n* points' for 'higher' values of *n*.

Nevertheless it is interesting to see that each of the above two sets is a model of the following axiom system:

I. for each two distinct points, there are exactly two circles containing them both
II. through each two distinct points not on a given circle there is exactly one circle which does not meet the given circle
III. not all points are on the same circle
IV. there exists at least one circle
V. every circle contains exactly three points

Notice that it is not the case that for each three distinct points, there is exactly one circle containing them all. We can achieve this by uniting the two sets to a model of a different axiom system:

I. for each two distinct points, there are exactly *four* circles containing them both
II. through each two distinct points not on a given circle there is exactly one circle which does not meet the given circle
III. not all points are on the same circle
IV. there exists at least one circle
V. every circle contains exactly three points

VI. for each three distinct points, there is exactly one circle containing them all

It is clear that we cannot draw a traditional picture of this model with twenty circles and six points that is as perspicuous as Figure 8. Already the ten circles of the first set present difficulties.

Let us now see what happens when we postulate that every circle contains exactly four points. It is clear that we need at least eight points and this means that we can proceed from a circle with seven points on it and a centre that joins them. Nothing is easier than to use the same partition of 7 as that of the corresponding projective geometry:

$$7 = 1 + 1 + 2 + 3$$

and add the centre of the circle to the slightly modified picture of Figure 10 (Figure 17):

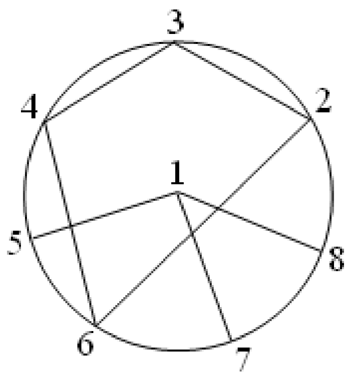

Figure 17

2346	1578
3457	1682
4568	1723
5672	1834
6783	1245
7824	1356
8235	1467

We are now ready to formulate the axiom system which has this set of circles as a model:

I. for each two distinct points, there are exactly two circles containing them both

II. through each two distinct points not on a given circle there is exactly one circle which does not meet the given circle; let us call the latter circle ‘separate’ from the former and conversely, and both circles ‘separate’ from each other

III. not all points are on the same circle
IV. there exists at least one circle
V. every circle contains exactly four points
VI. for each three distinct points, there is exactly one circle containing them all

Here we have all the ‘standard’ axioms, but we have also:

VII. for each two distinct circles that are not separate from each other, there are exactly two points contained by both

It is useful to discern such special axioms, as long as we have no uniform theory for affine circle geometries. In the end, we hope to have a set of axioms that enables us to formulate interesting problems about the existence or non-existence of models, for we cannot expect that every axiom of the form ‘every circle contains exactly n points’ can be satisfied.

At first sight, one might think that the partition of 11 that was used in projective circle geometry might help us to find a model for an affine circle geometry with the property that every circle contains exactly five points:

$$11 = 1 + 1 + 2 + 3 + 4$$

‘Just add a twelfth point and determine for the circle 1 2 3 5 8, or, in general for every circle of the form x, $x + 1$, $(x + 1) + 1$, $((x + 1) + 1) + 2$, $(((x + 1) + 1) + 2) + 3$, a circle containing the point 12 that is apart from it.?

However, none of the 15 combinations gives the desired result. For example, 7 9 10 11 12, which is apart from 1 2 3 5 8, has three points in common with 9 10 11 2 5 and similar violations of the axiom that there is at most one circle through three distinct points occur with the other combinations.

Nevertheless we stick to the method of transposition by looking for suitable partitions. Fortunately I found among the different partitions of 12 one special one that provided me with a model for such an affine circle geometry:

$$12 = 1 + 1 + 2 + 5 + 3$$

What makes it so special is that it is ‘incomplete’ in the sense that not every number under 12 is the outcome of two partial sums: 6 and only 6 does not occur as a sum in this partition. What this implies appears from the following numerical model generated by the partition:

1	2	3	5	10
2	3	4	6	11
3	4	5	7	12
4	5	6	8	1
5	6	7	9	2
6	7	8	10	3
7	8	9	11	4
8	9	10	12	5
9	10	11	1	6
10	11	12	2	7
11	12	1	3	8
12	1	2	4	9

Inspection reveals that there is for every circle exactly one circle that is apart from the given circle:

1	2	3	5	10	7	8	9	11	4
2	3	4	6	11	8	9	10	12	5
3	4	5	7	12	9	10	11	1	6
4	5	6	8	1	10	11	12	2	7
5	6	7	9	2	11	12	1	3	8
6	7	8	10	3	12	1	2	4	9

However, it is not the case that there is exactly one circle for each two distinct points not on a given circle which is apart from the given circle. There is neither any circle at all through 6 and 12, nor through 7 and 1, in general not through the pairs of points 1 7, 2 8, 3 9, 4 10, 5 11, 6 12. Neither the above axiom I, nor axiom II is satisfied by this set of circles.

But now look at the following picture (Figure 18):

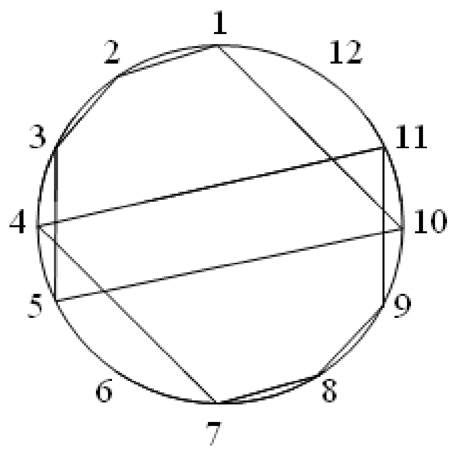

Figure 18

Nothing is easier to imagine than that the points 6 and 12 lie outside the disjunct circles 1 2 3 5 10 and 7 8 9 11 4. This can be used when we make an attempt to draw the first four circles as if they were part of a Euclidean plane (Figure 19):

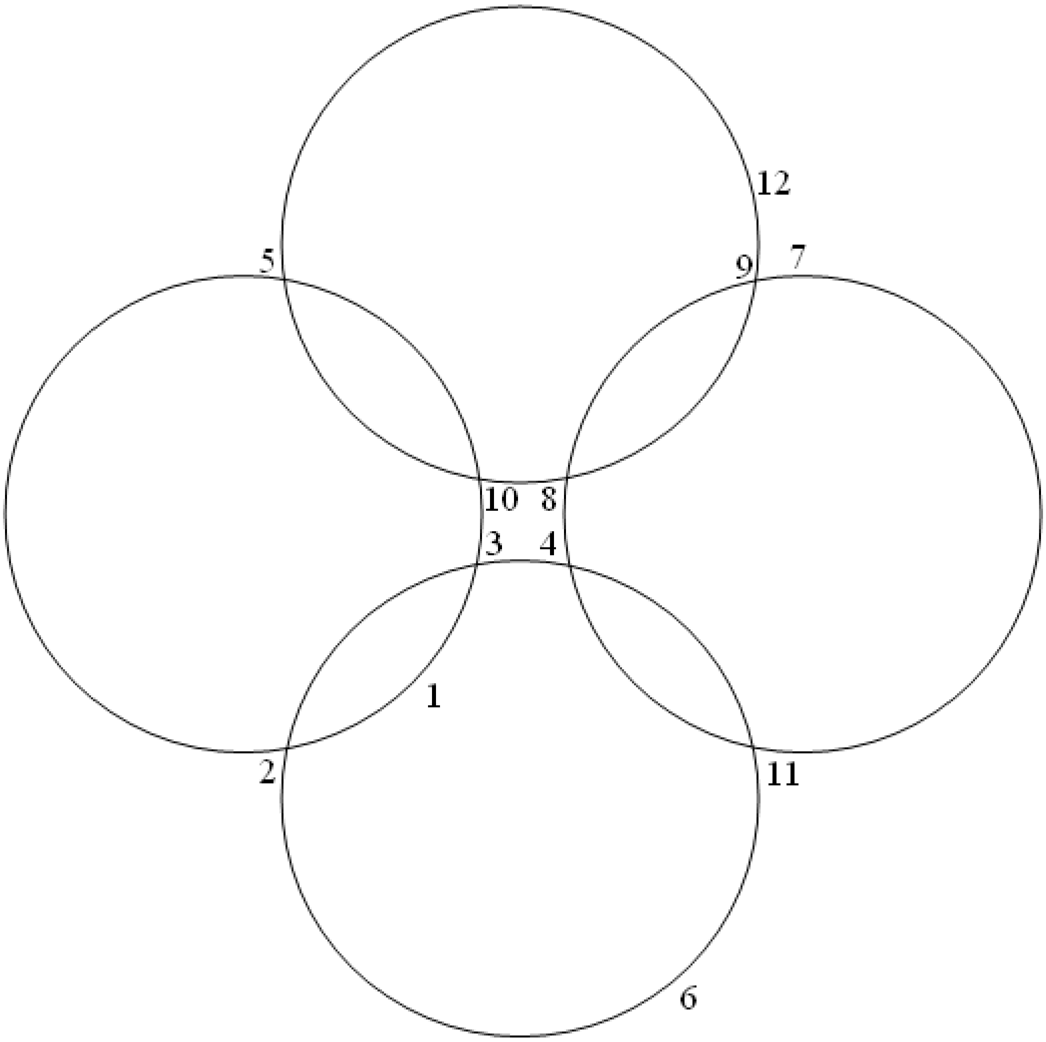

Figure 19

Then we can even add the fifth circle, 3 4 5 7 12 (Figure 20):

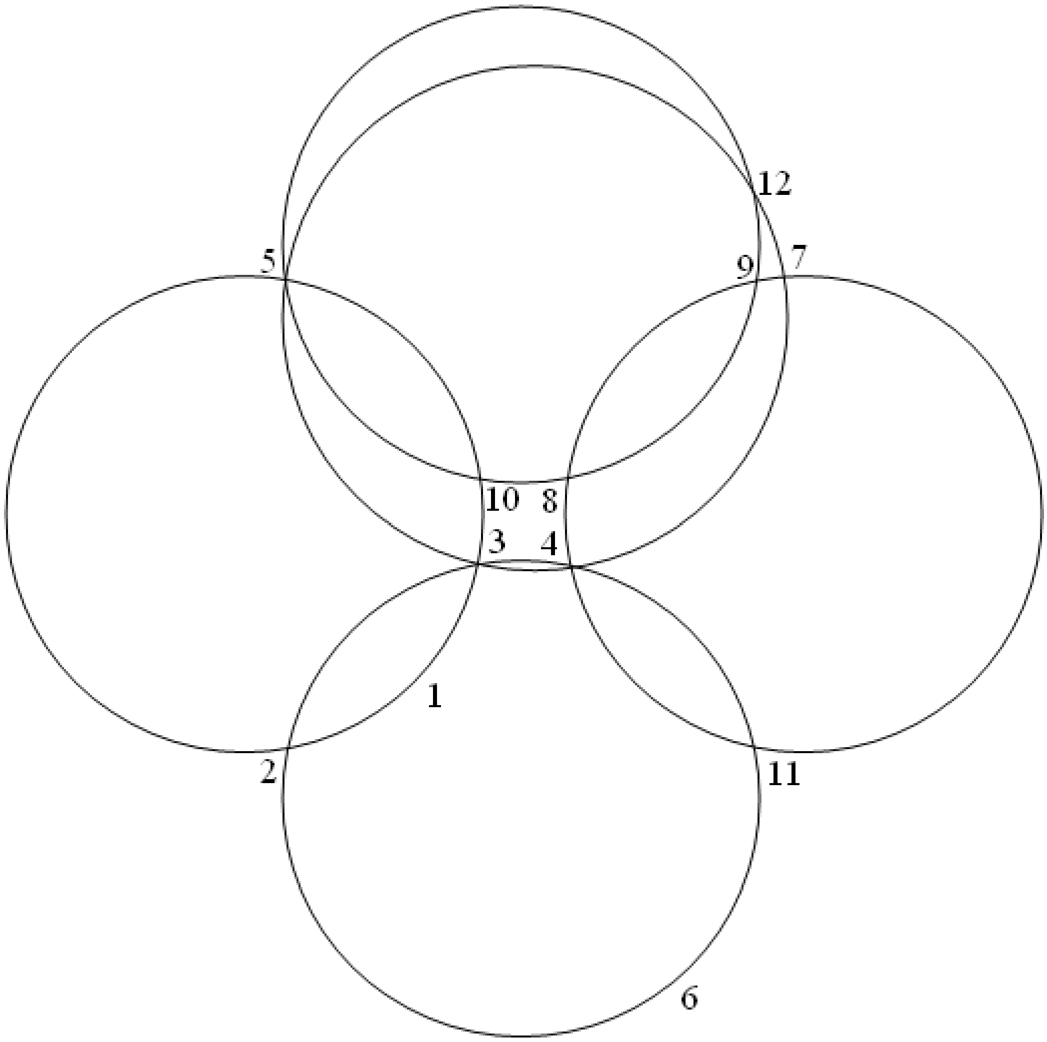

Figure 20

Similarly, the sixth circle, 9 10 11 1 6, could be added, albeit in the form of an oval figure, and so on.

The existence of such a simple model suggests that we should look for similar incomplete partitions as the above partition of 12, and preferably partitions with relatively few holes. As usual, ‘the computer’ helped us to find models which our ‘intuition’ could not give us. It appeared, again thanks to Dr. Donkers’ program, that there is no number that can be written as a sum of six numbers with that property. This dashes our hopes to find an affine circle geometry with the axiom that every circle contains exactly six points. But then there was the number 24, with the following partitions with seven components:[6]

$$24 = 11 + 2 + 3 + 1 + 4 + 2 + 1 \text{ with holes 9 and 15}$$

[6] The partitions are presented with the largest number first, in agreement with the print out of the computer.

24 = 11 + 2 + 1 + 4 + 2 + 3 + 1 with holes 8 and 16
24 = 10 + 3 + 4 + 2 + 3 + 1 + 1 with holes 8 and 16
24 = 10 + 2 + 2 + 1 + 3 + 5 + 1 with holes 7 and 17
24 = 9 + 6 + 1 + 3 + 2 + 2 + 1 with holes 11 and 13
24 = 9 + 3 + 3 + 2 + 5 + 1 + 1 with holes 4 and 20
24 = 9 + 1 + 6 + 2 + 3 + 2 + 1 with holes 4 and 20
24 = 7 + 5 + 2 + 4 + 4 + 1 + 1 with holes 3 and 21
24 = 7 + 3 + 6 + 4 + 2 + 1 + 1 with holes 5 and 19
24 = 5 + 5 + 3 + 3 + 4 + 2 + 2 with holes 1 and 23

The first partition leads to an affine geometry with an axiom that is again different from the standard one, according to which there would be for each two distinct points not on a given circle exactly one circle which is apart from the given circle. Consider, for example, the first circle of the first partition: 1 12 14 17 18 22 24. After nine rotations the tenth circle becomes: 10 21 23 2 3 7 9, and after fifteen rotations we get: 16 3 5 8 9 13 15. Both circles contain the points 3 and 9, and both are apart from the first circle.

Similar conclusions hold for the other partitions, only the second and the third partitions deserve special attention, because of the periodical character of the holes. For example, the second partition, 24 = 11 + 2 + 1 + 4 + 2 + 3 + 1, generates the following circles:

1 12 14 15 19 21 24	9 20 22 23 3 5 8	17 4 6 7 11 13 16
2 13 15 16 20 22 1	10 21 23 24 4 6 9	18 5 7 8 12 14 17
3 14 16 17 21 23 2	11 22 24 1 5 7 10	19 6 8 9 13 15 18
4 15 17 18 22 24 3	12 23 1 2 6 8 11	20 7 9 10 14 16 19
5 16 18 19 23 1 4	13 24 2 3 7 9 12	21 8 10 11 15 17 20
6 17 19 20 24 2 5	14 1 3 4 8 10 13	22 9 11 12 16 18 21
7 18 20 21 1 3 6	15 2 4 5 9 11 14	23 10 12 13 17 19 22
8 19 21 22 2 4 7	16 3 5 6 10 12 15	24 11 13 14 18 20 23

The upper circles of the first row, 1 12 14 15 19 21 24, 9 20 22 23 3 5 8, and 17 4 6 7 11 13 16 have now no points in common, but none of them contains the points 2 or 10 or 18. It follows that it is not the case that there is exactly one circle through each two distinct points not on a given circle which is apart from the given circle, although there is never more than one such a circle.

By inspection, we notice that none of the circles contain the pairs of points 1 9, 2 10, 3 11, 4 12, ..., 16 24, 17 1, 18 2, ..., 24 8 and hence we conclude that it is not the case that for each two distinct points there are exactly two circles containing them both. In this respect the affine circle geometry with 24 points is similar to that with 12 points. The question is how to characterize in geometrical terms the set of pairs of points that violates the axioms I and II. The situation is clear: for every two circles that are separate from each other, there is for every point contained by the first circle precisely one point contained by the other circle, such that both points are not together contained by any circle at all. We may call such points ‘unconnected’. Thus there is a certain duality between seaparate circles and unconnected points. In this respect, this model is similar to the model that resulted form the partition of 12, 12 = 1 + 1 + 2 + 5 + 3. Is this incidental?

The next incomplete partitions found by Dr. Donkers occurred with 40:

40 = 14 + 7 + 2 + 4 + 5 + 3 + 3 + 1 + 1 with holes 10, 20, and 30
40 = 13 + 6 + 2 + 1 + 2 + 4 + 7 + 1 + 4 with holes 10, 20, and 30

Apparently each of these partitions leads to four sets of circles. The first members produced by the first partition, are, respectively,

1, 15, 22, 24, 28, 33, 36, 39, 40
11, 25, 32, 34, 38, 3, 6, 9, 10
21, 35, 2, 4, 8, 13, 16, 19, 20, 21
31, 5, 12, 14, 18, 23, 26, 29, 30, 31

None of them contains the points 7, 17, 27, and 37 and we expect that the pairs of unconnected points are 1 8, 2 9, 3 10, ..., 40 10.

I conclude that the three sets of circles and points, produced by the ‘periodical’ partitions of, respectively, 12, 24, and 24, are a sufficient basis for the definition of an affine circle geometry, with the axiom scheme that every circle contains exactly *n* points. Until now we have models for the values 5, 7, and 9 of *n*. The question is only whether the axiom scheme has models for larger values ...

As long as we have no general theory, we are dependent on ‘the computer’ and I was delighted when Dr. Donkers informed me that 60 has four different partitions with double sums, to wit:

60 = 21 + 4 + 2 + 1 + 4 + 9 + 1 + 5 + 3 + 8 + 2 with holes 12, 24, 36, 48
60 = 16 + 1 + 8 + 14 + 5 + 2 + 4 + 4 + 3 + 2 + 1 with holes 12, 24, 36, 48
60 = 15 + 2 + 11 + 7 + 9 + 5 + 1 + 4 + 3 + 1 + 2 with holes 12, 24, 36, 48
60 = 11 + 6 + 11 + 4 + 10 + 8 + 1 + 4 + 2 + 1 + 2 with holes 12, 24, 36, 48

It follows that there is also an affine circle geometry with the property that every circle contains exactly 11 points. Can we proceed in the same way? That is to say, is there an affine circle geometry with the property that every circle contains exactly 13 points? Looking for periodical partitions of 84 in 13 parts is still a task that we may allot to the computer, but it it is clear that a general theory is badly needed.

Finite curve geometries?

The defining property of projective circle geometries is that the circles have precisely two points in common. This required that every partial sum of the number of points ocurrred twice. Affine circle geometries obey similar restrictions. It is clear that we can also ask for partitions in which every partial sum occurs three, or four, or even more times. For example, with five points the following partition leads to five geometrical forms which have precisely three points in common with each other (Figure 21):

$$5 = 2 + 1 + 1 + 1$$

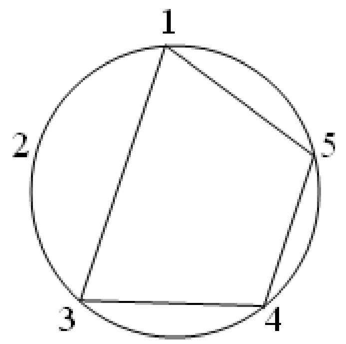

Figure 21

Figure 21 shows one of these forms in the shape of a quadrangle, but it is obvious that we can also think of curves (Figure 22):

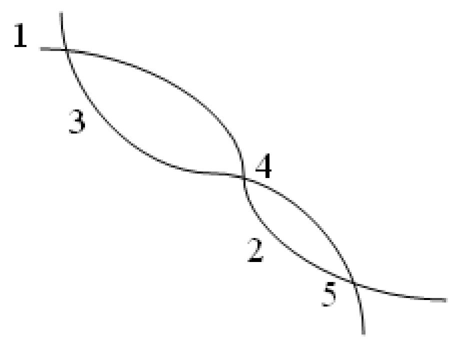

Figure 22

In Figure 22, two curves are drawn, one through 1, 3, 4, 5 and the other through 2, 4, 5, 1, and one has to imagine that there are also such curves through 3, 5, 1, 2 and 4, 1, 2, 3 and 5, 2, 3, 4.
Searching for complete partitions for such curve geometries is an exercise that can be best entrusted to a computer. Below are some results found by another computer program written by Dr. Donkers.

The following partitions for projective curve geometries are such that every two curves have three points in common:

$$11 = 4 + 2 + 1 + 2 + 1 + 1$$
$$15 = 5 + 2 + 3 + 1 + 2 + 1 + 1$$

There are also partitions for projective curve geometries such that every two curves have four points in common, for example:

$$15 = 4 + 3 + 1 + 2 + 2 + 1 + 1 + 1$$
$$19 = 5 + 2 + 2 + 1 + 1 + 1 + 3 + 3 + 1$$

Moreover, interesting partitions for projective curve geometries such that every two curves have five points in common were found with 19 and with 23, and partitions for projective curve geometries such that every two curves have six points in common with 13 and with 23.

An example of an affine curve geometry such that every two circles have at most three points in common is given by the following partition:

$$16 = 6 + 3 + 1 + 1 + 2 + 2 + 1$$

Its hole is ‘at’ 8, so there are two sets of circles, and it is easy to see that their first members are 1, 7, 10, 11, 12, 14, 16 and 9, 15, 2, 3, 4, 6, 8 respectively.

It is difficult to tell how significant such structures are. Will it be possible to prove general theorems about them? The answer lies in the future.