ARISTARCHOS' VISIT TO EUCLID

SERVANT. Master, there is a man outside who wants to speak to you. He says that he is a philosopher.

EUCLID. Please, let him or her come in!

ARISTARCHOS (*entering Euclid's room*). Good afternoon, my name is Aristarchos, I live in Athens, but I came to Alexandria in order to read your papers on geometry. I just went through part one of your *Elements*, but I have so many questions about it that I venture to visit you. Am I welcome?

EUCLID (*shaking hands with Aristarchos*). That's a surprise! I have always wished to see you since I read your writings. We have several of them in our library. But you are not going to tell me that you had difficulties following my proofs, are you?

ARISTARCHOS. No, that is not my problem; my questions are deeper, so to say.

EUCLID. Oh, that's why you call yourself a philosopher! But enough compliments, let us get to work! Tell me what bothers you.

ARISTARCHOS. Well, when I reached your propositions 20 and 21 – do you know which I mean?

EUCLID. Of course! I even ask my students to learn the numbers of the propositions by heart…

ARISTARCHOS. I learned them too, if only in order to memorize the order in which they are proved. I know, for example that Proposition 21 says that (*Aristarchos speaks as if he is teaching*) **if on one of the sides of a triangle, from its extremities, there are constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.** (*Aristarchos draws the following figure in Euclid's sandbox*):

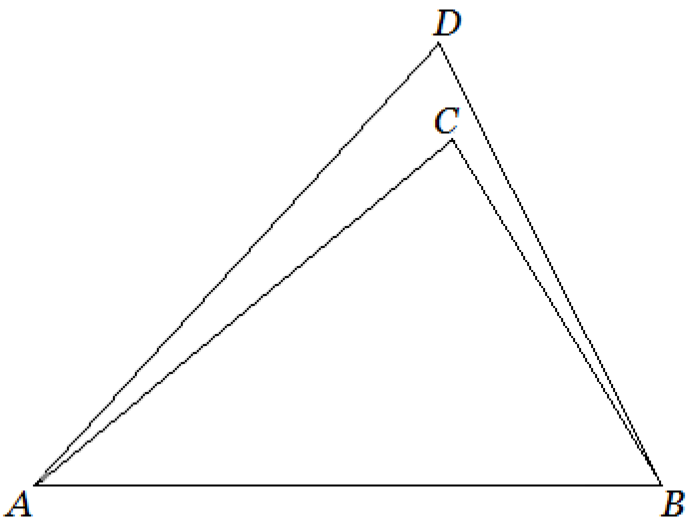

Figure 1

And this brings me to the following questions: why can this proposition only be proved after proposition 20 and why is proposition 20 itself proved as late as it is? Is it not easy to see that (*emphatically*) **in any triangle two sides taken together are greater than the remaining one**? One can even imagine that a philosopher, though not myself, would think that it is such a fundamental insight that it could be regarded as an axiom...

EUCLID. It is true that it is a central proposition, as it imposes a restriction on the forming of triangles, to mention only one consequence. But believe me or not, just because I hate those philosophers who believe that some propositions are self-evident, whatever that may mean, or can be perceived by intuition, whatever that may mean, I deliberately decided to make them ridiculous by designing a system in which the proposition that (*with emphasis*) **in any triangle two sides taken together are greater than the remaining one** comes only after a long chain of reasoning. In fact, I started my research by asking what is really needed in order to prove this theorem!

ARISTARCHOS. That is remarkable, because our Athenian philosophers believe that you actually started with some self-evident principles, and then proceeded to prove a series of theorems one after another. But it is the other way around: your starting point was formed by well-known and useful theorems, and your attempts to prove them provided you with sub-theorems and, eventually, postulates. Am I right?

EUCLID. Quite so! One of my favourite examples is my proof of Pythagoras' theorem. This proof suggested the method with which I dealt with areas of triangles and quadrangles. But maybe we can talk about this another time and concentrate on part one. It is true that the 'triangle inequality,' as I call proposition 20, has a high status, similar to Pythagoras' theorem. That's why I built the first part of my Geometry around the triangle inequality theorem. Let me explain (*while he draws the following figure in the sandbox*):

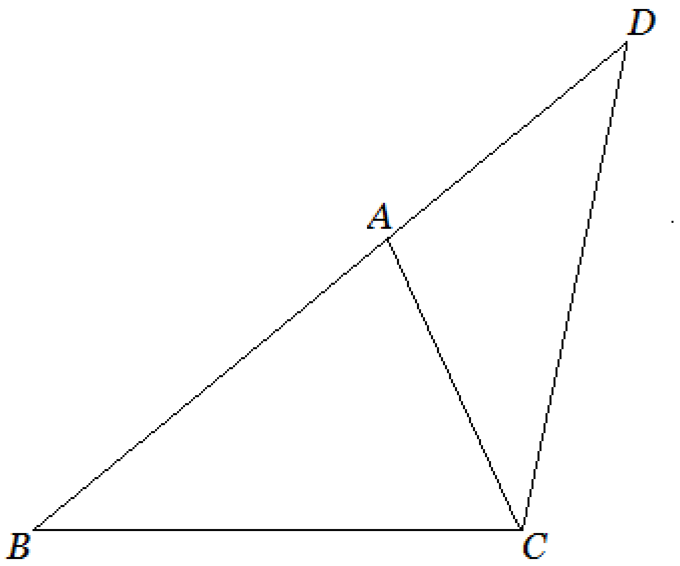

Figure 2

ARISTARCHOS. I see, you wanted to prove that the two sides *AB* and *AC* of triangle *ABC* taken together, are greater than the other side *BC*, but since you can only compare single lines, you took a point on the prolongation of *BA* such that *AD* is equal to *AC*, and you had to prove that *BD* is longer than *BC*.

EUCLID. Excellent! And that is why I needed a theorem that makes such a construction possible. But notice that I already drew a triangle, and moreover, prolonged one of its sides. I ensured that this is possible by my first two postulates:

> Postulate 1.
> To draw a straight line from any point to any point.
>
> Postulate 2.
> To produce a finite straight line continuously in a straight line.

Furthermore, in order to determine the point *D*, I needed the theorems that you know as proposition 2 and proposition 3, and I managed to prove them with only one preceding theorem, proposition 1. I shall mention them for regularity's sake.

> Proposition 1.
> To construct an equilateral triangle on a given finite straight line.
>
> Proposition 2.
> To place a straight line equal to a given straight line with one end at a given point.
>
> Proposition 3.
> To cut off from the greater of two given unequal straight lines a straight line equal to the lesser one.

We can discuss the corresponding proofs later, as you wish, but at least it must be clear to you that I had to justify the possibility of drawing auxiliary lines and circles. Therefore I adopted

> Postulate 3.
> To describe a circle with any center and radius.

ARISTARCHOS. This is all clear to me, and I immediately infer that your desired proof of proposition 20 must make use of the theorem of the isosceles triangle, applied to the triangle *CDA* in your figure.

EUCLID. Quite right, and this brought me to the next two theorems of my system. You know them:

> Proposition 4.
> If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
>
> Proposition 5.
> In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

ARISTARCHOS. I even know your proofs of these theorems, and I already begin to understand that these theorems and not proposition 20 appear in the beginning of your system. But now I am anxious to hear how your attempts to find a proof of proposition 20 itself resulted into the intermediate theorems. Were they all necessary for the solution of this problem?

EUCLID. No, they were not. Afterwards I inserted some theorems that are simple consequences of directly preceding theorems. But let us return to the figure for proposition 20. You already remarked that I had to prove that *BD* is longer than *BC*. It is important that these lines can be seen as sides of the triangle *BCD*, because we can now compare the angles of this triangle that lie opposite the sides *BD* and *BC*, namely the angles *BCD* and *BDC*.

ARISTARCHOS. Aha, the angle *BCD* consists of two parts, and one of them, the angle *ACD* is equal to the angle *ADC* of the isosceles triangle *CDA*, or what amounts to the same, the angle *BDC* of the triangle *BCD*. So we must conclude that the angle *BCD* is greater than the angle *CBD*. Now it suffices to prove that in any triangle the side opposite the greater angle is greater.

EUCLID. Stop, you are right, but your earlier conclusion that the angle *BDC* is greater than the angle *CBD* requires as much a justification as your conclusion about the equality of the angles *ACD* and *ADC*, for which you implicitly appealed to proposition 5 about isosceles triangles.

ARISTARCHOS. But the angle *BDC* consists of two parts, so it is greater than each of it parts, and therefore it is also greater than something that is equal to such a part.

EUCLID. Quite right. But we must explicitly mention this in our proof, just as we must refer to proposition 5 in order to justify that the two angles are equal.

ARISTARCHOS. But proposition 5 mentions a property of triangles, whereas the properties of wholes and parts and the properties of equalities are not so specific.

EUCLID. That is why I included them in my system under the head of “common notions”; you must have seen all nine, but I remind you of the first and the last one:

> Common notion 1.
> Things which equal the same thing also equal one another.
>
> Common notion 9.
> The whole is greater than the part.

Athenian philosophers would perhaps say that such principles speak for themselves, in other words are self-evident, but this opinion results only from the fact that principles like these are so often used in everyday reasoning, that everyone takes them for granted. Some of my students even wondered why I mention nine common notions, because they believed that some of them need not be mentioned at all, and that five will suffice. But I think that one cannot cautious enough to make one's assumptions as explicit as possible.

ARISTARCHOS. This is an extremely important remark. Do you mean that your common notions are not obvious?

EUCLID. Indeed they are not. I found them when I looked for proofs, such as that of the triangle inequality theorem, so they were not beforehand clear to me. The nine common notions that are part of my system concern geometrical things such as points, lines, angles, and other figures, and also areas and volumes, that is to say, in so far the mentioned properties are applicable to them.

ARISTARCHOS. Does this mean that your common notion 9 does not hold for points?

EUCLID. That is correct. And the conclusion must be, *pace* your Athenian colleagues, that the Common notions are just postulates which can be used in geometrical proofs. Nothing has been said about other applications. Moreover the fact that they might be used in everyday reasoning about other things than geometrical objects is not a valid argument for regarding them as evident principles. But I have already said too much about your colleagues, for I don't give a cent for their idle talk. Let us once more return to the proof of proposition 20. You were right to notice that it is sufficient for our purpose to prove that in any triangle the side opposite the greater angle is greater. However this is easier said than done, as you may have inferred from the procedure in my book. I did not find a direct proof of the preceding proposition 19 that indeed says that in any triangle the side opposite the greater angle is greater.

ARISTARCHOS. I know, and I wondered why you gave an indirect proof of proposition 19. As far as I can see, it is merely a logical consequence of proposition 5 and proposition 18 that says that in any triangle the angle opposite the greater side is greater. You simply assumed that in your figure the side opposite the greater angle is not greater, and it appeared that this is excluded by proposition 5 and proposition 18 together. What I miss is a direct insight into the content of proposition 19.

EUCLID. Well, to ask for direct insight is asking too much. Given the way in which I proved this theorem, we do not even have a direct insight into the content of proposition 2. You may have heard of the misgivings of some of my colleagues about my proof.

ARISTARCHOS. Yes, but that is not my problem. I admire your proof of proposition 2. It is purely geometrical, and what else do we want? Perhaps I incorrectly used the philosophical terminology of my Athenian colleagues when I asked for a direct insight. What I meant was that your indirect proof of proposition 19 establishes no geometrical connection between the size of an angle and the opposite side of a triangle. You made as it were a logical detour in order to gain the desired result and this gives me a certain feeling of discomfort.

EUCLID. I share your opinion, but unfortunately I saw no other way of building my system. Besides I rather easily found a proof of proposition 18 that establishes a connection between the size of a side and the opposite angle of a triangle, in this order. That is to say, it is true that I still needed some intermediate theorems, but eventually everything came down to the propositions 4 and 5, if I leave the propositions 1 and 3 out of consideration. As you know, these theorems are required for the auxiliary lines in the figures. Did you notice how strong the propositions 4 and 5 are when you studied part one?

ARISTARCHOS. I did, but nevertheless I would like to know how you discovered that the curious proposition 16 helps to prove proposition 18, and what is more, how you found your amazing proof of proposition 16. You see that I already memorized the number of this important theorem that says, let me see (*with emphasis)*: **In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.**

EUCLID. You are asking two questions. In order to answer them, I will continue my explanation by analysing proposition 18 (*draws the following figure*).

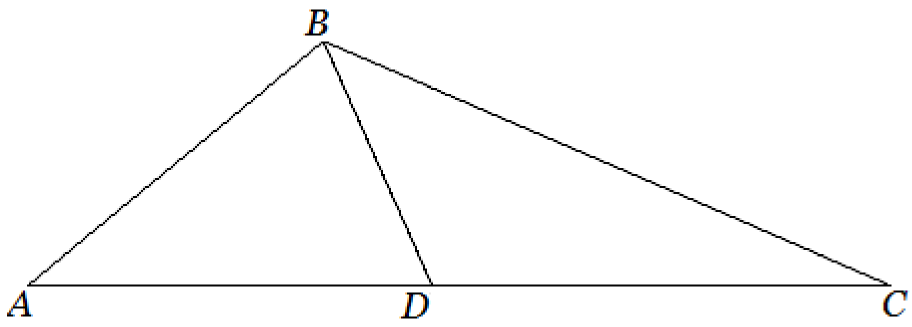

Figure 3

ARISTARCHOS. I see, you already made use of the assumption that *AC* is greater than *AB*, by determining the point *D* on *AC* such that *AD* is equal to *AB*. Now you wanted to prove that the angle *ABC* of triangle *ABC* is greater than the angle *ACB*. Fortunately the angle *ABC* is a whole consisting of two parts, so you decided to prove that the angle *ABD* is at least as great as the angle *ACB*. The angle *CBD* is not considered for such an equation, because it is easily seen that this angle can be greater than the angle *ACB*. This is already the case in your figure. But *ABD* is an isosceles triangle, and it follows that the angle *ABD* is equal to the angle *ADB*. This means that your task came down to proving that the angle *ADB* is at least as great as the angle *ACB*. But how did you find out that there must be a theorem such as proposition 16?

EUCLID. The idea came to me when I looked at the triangle *DCB* and suddenly saw that the angle *ADB*, which I regarded as one of the interior angles of the triangle *ABD*, can also be seen as an exterior angle of the triangle *BDC* (*Euclid draws a small arc in his last figure*).

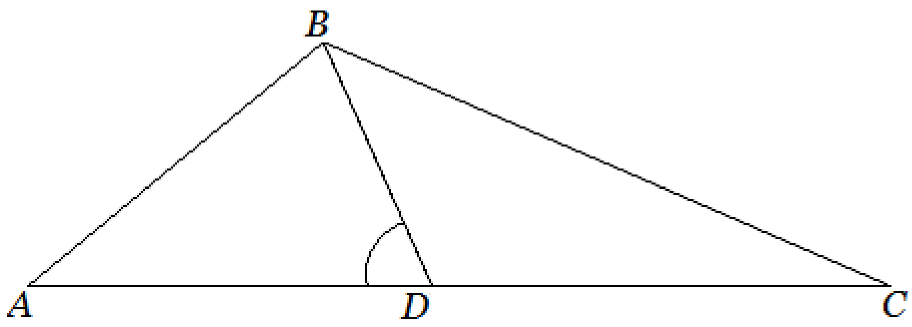

Figure 4

ARISTARCHOS. That is interesting. The angle that you marked just now was described by you in two ways, and each of them corresponded with a particular way of seeing. These two perspectives cannot be taken simultaneously; it requires a changeover to pass from the one to the other. Did you really have such a sensation?

EUCLID. Yes, and you and I can repeat it by first focussing on the left side of the figure, and then on its right side. But what is also important, I intuitively inferred in the second case that the exterior angle *ADB* of the triangle *BDC* is greater than its interior angle *DCB* and I almost simultaneously drew the promising conclusion that this is a general property of triangles. In other words, in any triangle, if one of the sides is produced, then the exterior angle is greater than either of the remote interior angles. There you are.

ARISTARCHOS. But I thought that you did not believe in intuition. You reproached my Athenian colleagues that they appeal to intuition, and now you confess that you drew intuitive inferences yourself. What is this?

EUCLID. The difference is that they believe that intuition can give us knowledge, whereas I see it only as a spontaneous discovery of something conspicuous during the search for a solution of a problem. Such intuitive inferences remain sometimes isolated in the sense that they do not contribute to a solution, but it also happens that they are followed by a promising conclusion that points to a certain direction and stimulates you to do further work on your problem. This was the case when I tried to prove proposition 18, and I immediately set myself to find out what would be required to establish the theorem of the exterior angle.

ARISTARCHOS. That is indeed completely different from the Athenian view of intuition. Your promising conclusion stood only at the beginning of a presumably long process of finding intermediate theorems. I do not think that my Athenian colleagues could have done what you did, bringing your task to a good end, without new postulates. Please tell me now how you filled up the gap between proposition 16 and proposition 5. How did you deal with the extremely important intermediate theorem 15, which says that two intersecting straight lines make the opposite angles equal to one another? It looks so simple when we look at the figure, but apparently this is an illusion. Back home I shall try to convince my colleagues that they are wrong to believe in intuition as they do. I hope that this example will convince them. (*Aristarchos draws the following figure in the sand*)

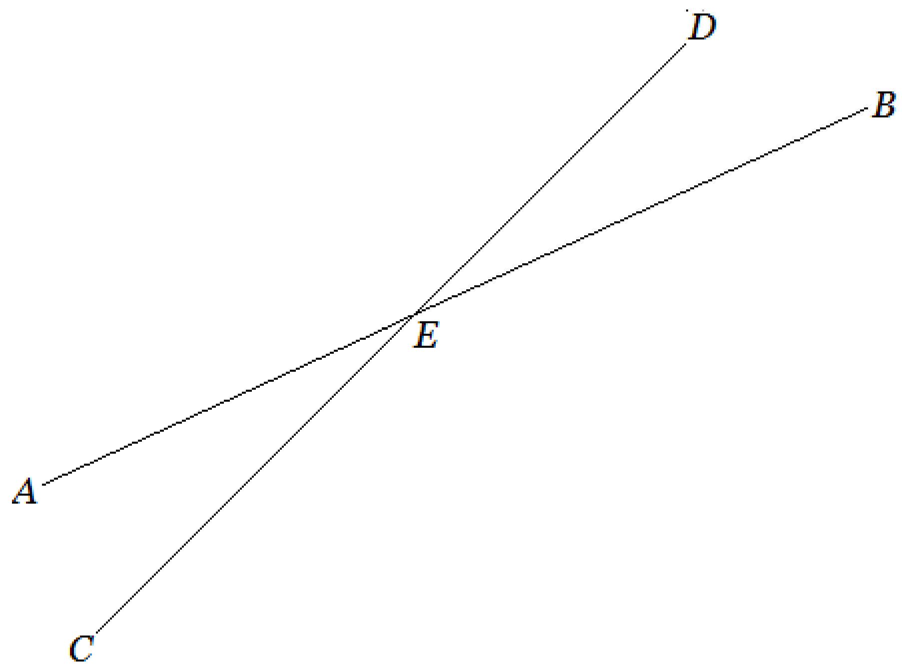

Figure 5

EUCLID. This is indeed a fine example. Your colleagues may call it evident that the angles *AEC* and *BED* are equal, but if they do this, then they have not understood one jot of my approach. But let me now tell you how I proceeded with proposition 16. After that, I will comment on proposition 15. I will make use of the figure for proposition 18, but produce the line *BD* further to get the line *BE* (*does this with the following result*):

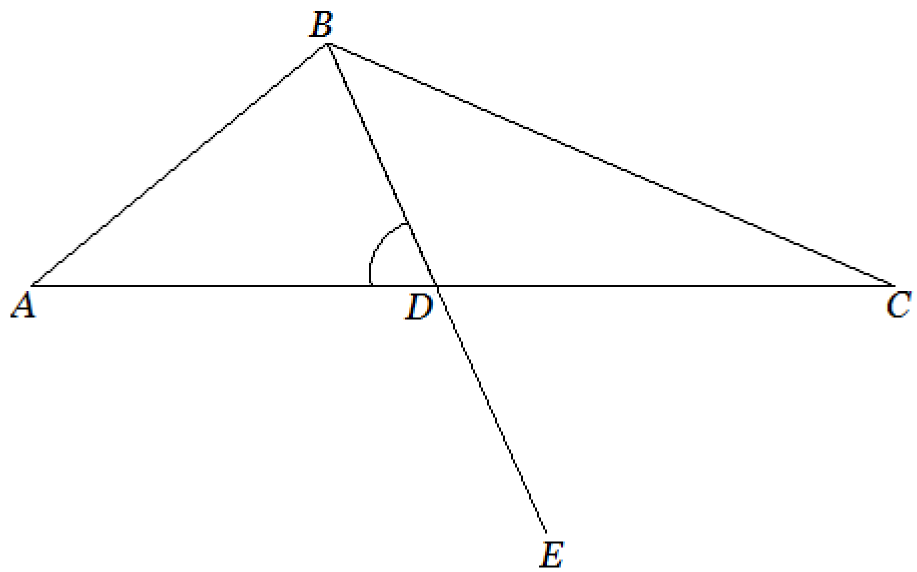

Figure 6

The problem was that I wanted to prove that the exterior angle *ADB* of the triangle *BDC* is greater than its interior angle *BCD*, but this effort failed, whereas it succeeded with the exterior angle *ADE*. Look, if this angle is indeed greater than the angle *ACB*, then it must contain as a part an angle that is at least as great as the angle *ACB*. Now I imagined that it would contain an angle that is equal to the angle *ACB* (*draws one more line in the figure*):

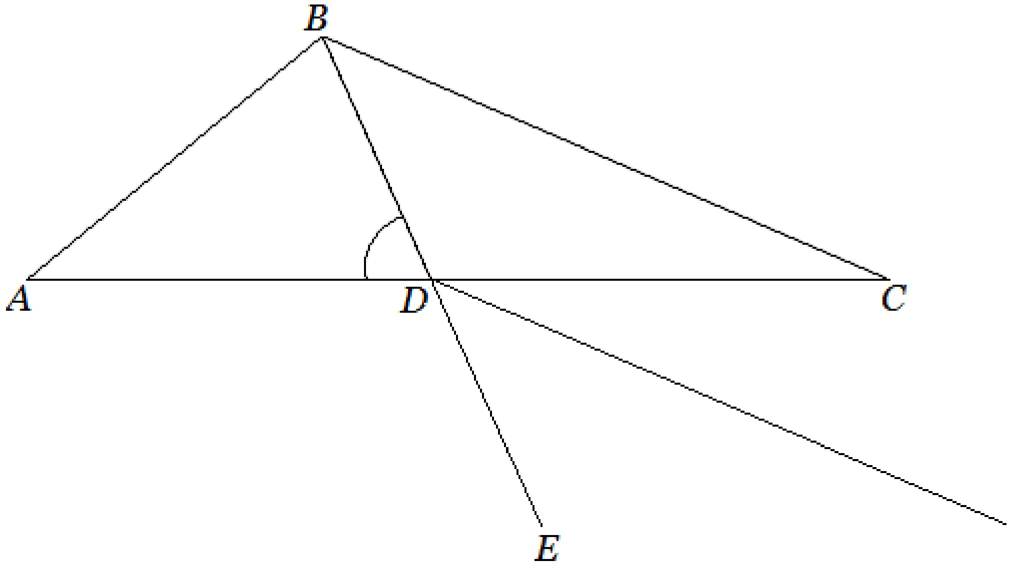

Figure 7

But how do we get a point *F* on this new line such that the angle *FDC* is equal to the angle *BCD*?

ARISTARCHOS. I know the answer, because I already saw your proof, but I presume that you argued that equal angles must come from an application of proposition 4. Therefore you looked for two triangles that not only have two sides equal to two sides respectively, but also have the angles contained by the equal straight lines equal. I am impressed that you got these triangles by first determining the middle *M* of the line *CD*, and then producing *BM* further to get a point *F* such that *MF* is equal to *MB*. (*Aristarchos draws a new figure*).

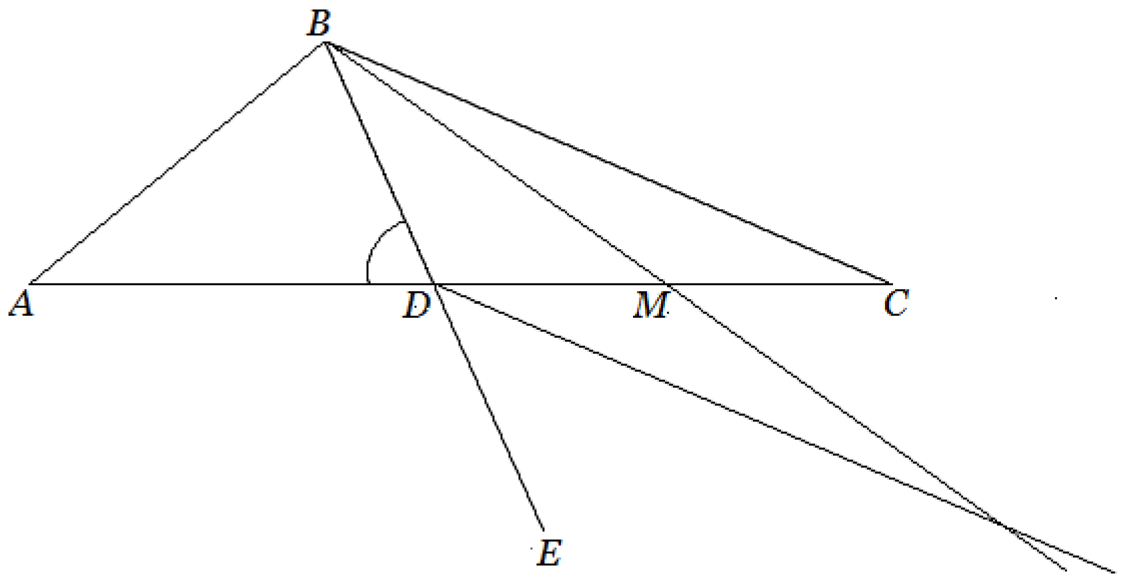

Figure 8

EUCLID. Your answer does indeed describe the way in which I found the proof, but I must confess that I got the idea of the point *M* in the middle of the line *DC* only after I considered that the required point *F* forms a parallelogram with the points *C*, *B* and *D* (*draws the line CG parallel with the line BE in his original figure and adds the letter F*):

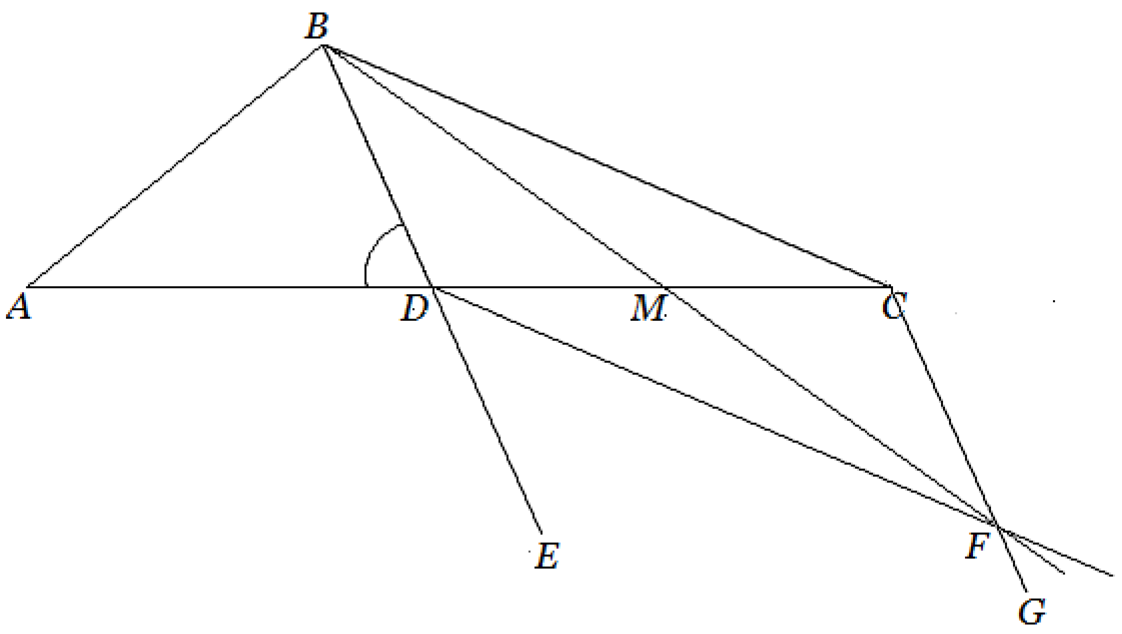

Figure 9

ARISTARCHOS. That is remarkable. But is it not strange that you use a result from a later part of your system in an earlier part?

EUCLID. It is true that I have no theorems about parallelograms in this part of my geometrical system, but I have no objections against the use of knowledge of later parts in order to find a proof in an earlier part, as long as it does not affect the proof itself. And it follows from your summary of the proof of proposition 16, that this is not the case here.

ARISTARCHOS. I agree. It reminds me of a proof that I once found for the following interesting problem (*He draws the next figure*):

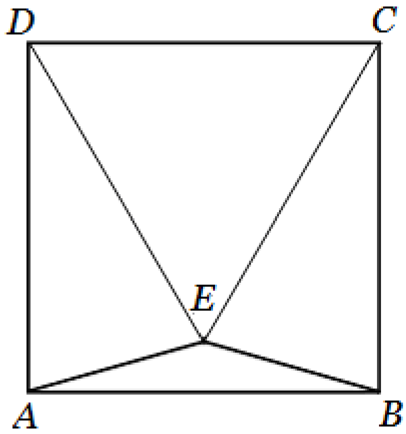

Figure 10

Suppose that the angles *EAB* and *EBA* are one sixth of a right angle. Prove that the triangle *CDE* is equilateral.

EUCLID. I know this problem, and my solution begins with the construction of a point *F* within the square such that the angles *FBC* and *FCB* are also one sixth of a right angle. But what is your solution?

ARISTARCHOS. Let me first tell you how I discovered my solution. Looking at the triangle *AEB*, I suddenly saw it as a chord triangle of an equilateral and equiangular twelve-angled figure. Look (*adds some lines to the figure*):

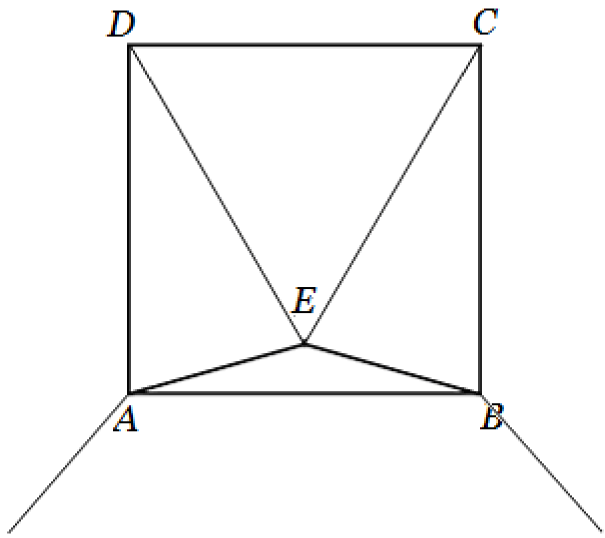

Figure 11

EUCLID. I see. Then you construed the center *H* of the circumscribed circle by bisecting the lines *AE* and *BE* and the rest is easy (*draws some more lines in Aristarchos' figure*).

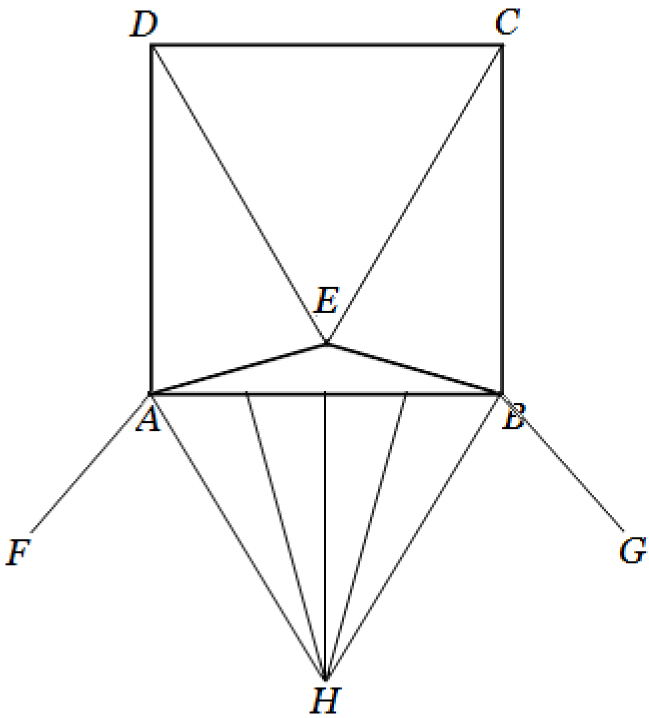

Figure 12

ARISTARCHOS. Yes, but I did not call *H* the center of this circle. Moreover I did not draw the line *AF* and *BG* in my proof. My students understood it, even though they had never heard of equilateral and equiangular twelve-angled figures.

EUCLID. That is fine. Yet it remains interesting that you and I saw things in our figures that are not there, so to say. But the resulting intuitive inferences did excellent work!

ARISTARCHOS. That is to say, their promising conclusions came true!

EUCLID. I think that we must distinguish between the way in which a solution is discovered and the solution itself. In my systematic approach there are high demands upon the proofs, but I kept the history of my discoveries hidden.

ARISTARCHOS. That is one of the reasons that I came to see you, and I am already much wiser. I understand your combined proof of proposition 16 and proposition 17, but I conclude once more how important proposition 15 is for this combined proof. You used it twice, first in order to reach the conclusion that the angles *BMC* and *FMD* are equal, and second in order to argue that the angle *ABD* is equal to the angle *CDE*. This makes me the more curious about your derivation of proposition 15. I saw that you needed a relatively large number of intermediate propositions, namely proposition 13, proposition 11, proposition 8 and proposition 7.

EUCLID. Good question. Let us look again at your figure for proposition 15:

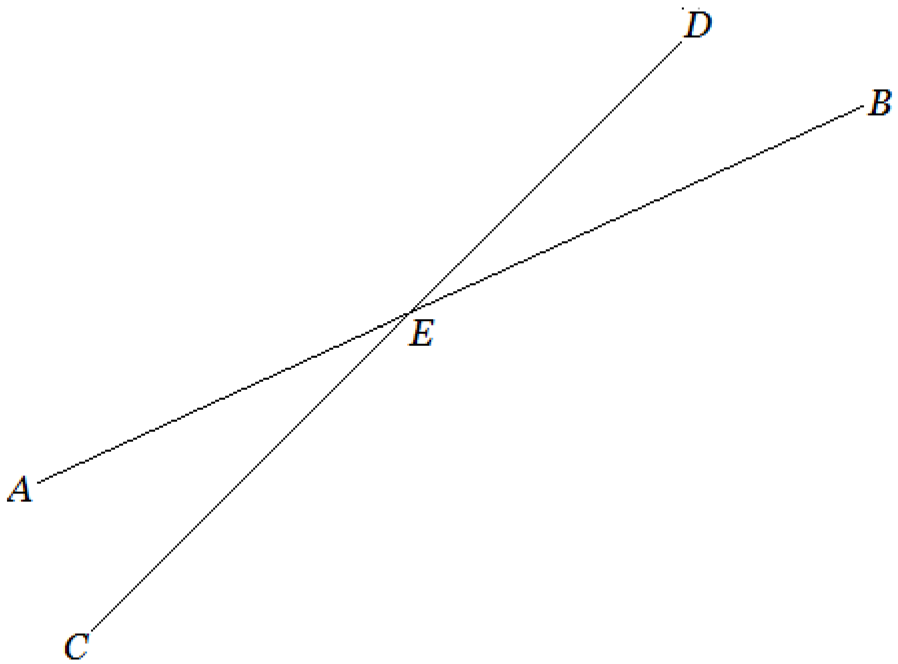

Figure 13

My problem was that I found it difficult to regard the figures *AEB* and *CED* as angles. To me they look more as flattened triangles. But if I would call them angles, then this would be of no help, for then they could not be angles of triangles. This means that proposition 4 is anyhow not applicable to them. Perhaps do you now understand why I defined angles as I did?

ARISTARCHOS. You mean:

> Definition 8.
> A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

EUCLID. Quite right. But after a long period of thinking about it, I saw that a proof of proposition 15 would not cause difficulties anymore, as soon as I could prove that the two angles *AED* and *BED* are together equal to two right angles.

ARISTARCHOS. Now I also begin to understand your

> Definition 10.
> When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

For this brought you to your

> Proposition 13.
> If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

EUCLID. That is correct. But do not forget

> Postulate 4.
> That all right angles equal one another.

ARISTARCHOS. I see. And the proof of proposition 13 requires that it must be possible to draw a straight line at right angles to a given straight line from a given point on it. (*Aristarchos draws the following figure.*)

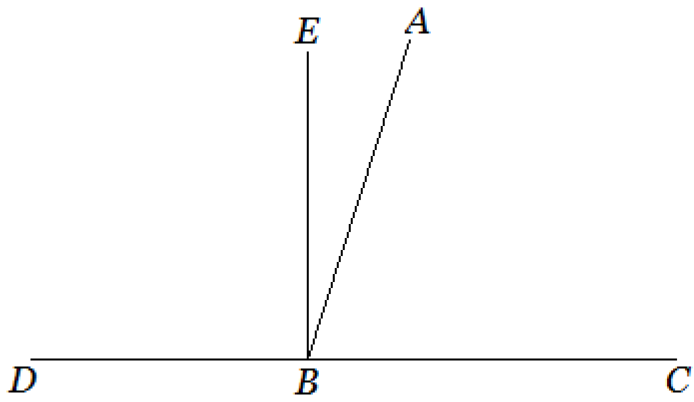

Figure 14

If the angle *CBA* equals the angle *ABD*, then they are two right angles, according to definition 10. But if not, then we draw *BE* from the point *B* at right angles to *CD*. This is the crucial step, for now the angles *CBE* and *EBD* are two right angles. The rest is easy with the help of your common notions.

EUCLID. Yes, and the possibility of the construction of the line *BE* is explained in proposition 11. The construction itself was simple, for I had the propositions 1 and 3 at my disposal. (*Euclid draws the following figure.*)

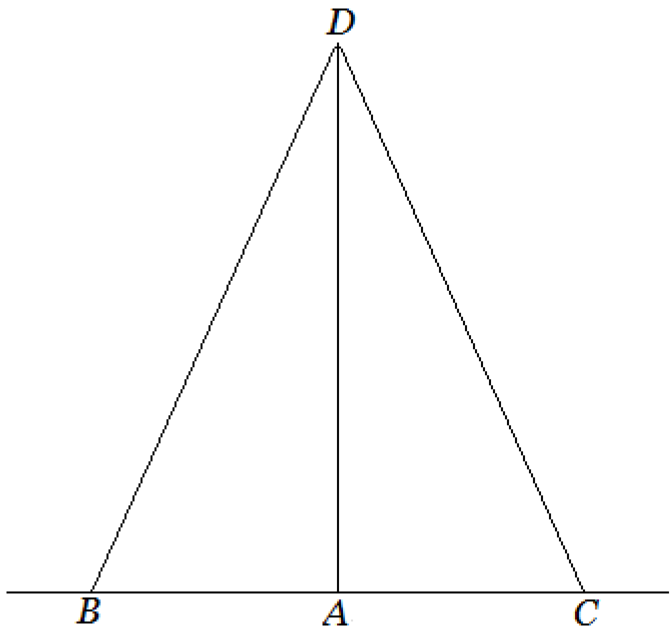

Figure 15

The only thing that gave me the greatest troubles was the proof of the theorem that guarantees that the triangles *ABD* and *ACD*, which have their three sides equal, have also their angles equal. I mean:

> Proposition 8.
> If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal, which are contained by the equal straight lines.

ARISTARCHOS. I have seen that your proof was a simple application of the preceding theorem. I know what it amounts to, namely that the construction of a triangle with two sides equal to two sides, respectively on the same base is unequivocal (*Aristarchos draws a new figure*).

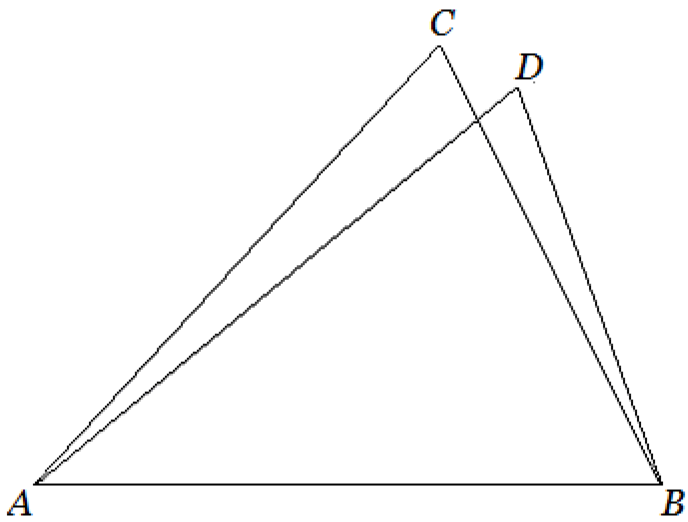

Figure 16

But your formulation of proposition 7 is so complicated that I could not memorize it literally. It states, in my words, that given a certain triangle with certain sides, it is impossible to construct another triangle at the same side on the same base and with two equal sides respectively in the same position. This means, in my figure, that if the lines *AC* and *AD* are equal, and if the same holds for the lines *BC* and *BD*, that then the points *C* and *D* coincide.

EUCLID. I have no problems with your description. Can you also prove this theorem with the help of proposition 5? When I found it, my original task of inventing a system in which the triangular inequality theorem can be proved, was accomplished. This makes that proposition 7 is, in a sense, my favourite theorem, though it is only used in the proof of proposition 8 and nowhere else. Moreover, I have a special feeling about my proof. It is a simple, but not a perspicuous proof, and I can use this fact as a weapon against philosophers who still think that proofs must give a clear reason why the theorem holds. Your criticism, that the formulation of this theorem is already intricate, is right. It is also peculiar that the conclusion is negative, for it is said about a certain construction that it cannot be executed. Now I ask you: can you imagine an impossible situation, ahem ... But the theorem could be proved, so there is an explanation for it and the explanation can be followed step by step. Yet I think that philosophers will still have misgivings about it, because they miss direct insight into both the content of the proposition and the proof of the proposition. And this is in sharp contrast with the direct insight that they contend to have into the theorem that appears in my system as late as proposition 20, the triangular inequality theorem.

ARISTARCHOS. I am flabbergasted. I hope that I can reproduce your proof of proposition 7 back in Athens, so that I can have a meaty discussion with my colleagues. If you allow me, I shall give your proof here and now, so that you can correct me if I go wrong.

EUCLID. Please, go ahead. I will try to react as if I am an Athenian philosopher, though I am not sure that I understand them well.

ARISTARCHOS. I am not afraid that your comments will be out of place, given your smartness. But I can not make myself answerable for my colleagues. But let me start. Suppose that *C* and *D* are different, as my figure already shows. I'm going to derive something that is logically impossible. As you know, this is not the end of the matter, because in my figure the point *D* lies outside the triangle *ABC*, and therefore I must give a similar reasoning for the case that *D* lies inside the triangle *ABC*. (*Aristarchos draws another figure.*)

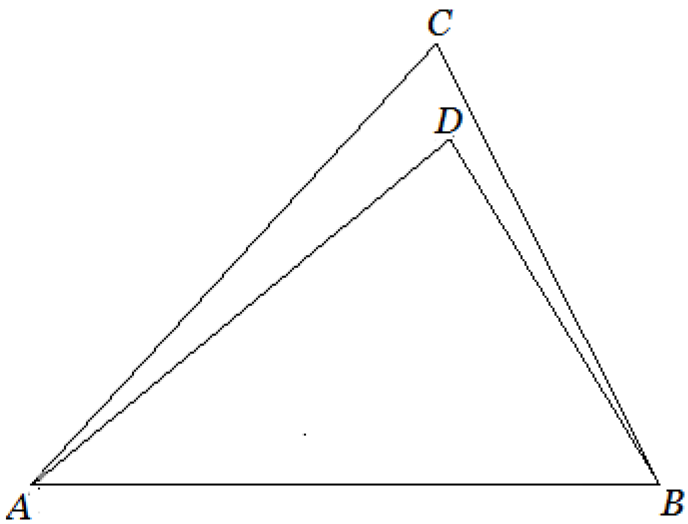

Figure 17

EUCLID. Are you sure that there are no other cases to consider?

ARISTARCHOS. Well, *D* lies outside or inside the triangle or else it coincides with *C* for it cannot lie elsewhere on the lines *AC* and *BC*. In the last case, there is nothing to prove any more.

EUCLID. I agree. But suppose now that *D* lies outside the triangle *ABC*. You said that you would derive a contradiction. But this implies that your figures show impossible situations. How is that possible?

ARISTARCHOS. I did not make the lines *AC* and *AD* and the lines *BC* and *BD* equal by performing a construction. I just assumed that *D* lies outside the triangle ABC, and then postulate 1 implies that there is a line *CD*, whereas all its points except *C* lie also outside the triangle *ABC*. This can be concluded without taking the special properties of the drawn figure into account. The figure does not show an impossible situation, because it does not take the equality of the lines into account.

EUCLID. This means that the figure is incorrect. How can you be so sure that this fact does not influence your reasoning?

ARISTARCHOS. Because I support each step by a reference to a definition, a postulate, a common notion, or an already proved proposition.

EUCLID. OK. Go on.

ARISTARCHOS. Look at my first figure. I told you that there is a line *CD*, so let me draw this line (*Aristarchos connects the points C and D in his first figure.*)

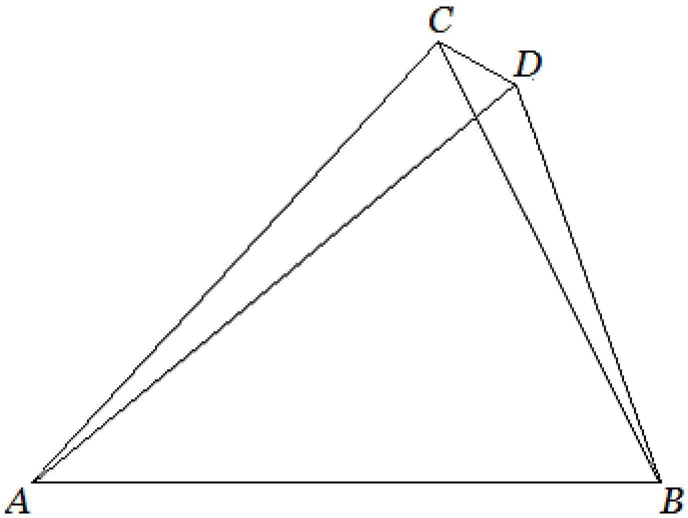

Figure 18

It follows that the triangles *ACD* and *BCD* are isosceles triangles, so we can apply proposition 5. In particular, the angle *ADC* is equal to the angle *ACD*. Therefore, the angle *ADC* is greater than the angle *DCB* that is a part of the angle *ACD*. It follows that the angle *BDC*, which contains the angle *ADC* as a part, is much greater than the angle *DCB*. But according to proposition 5 the angle *BDC* is equal to the angle *DCB*. Now it is demonstrated that it is both much greater and equal and that is impossible. Consequently *C* and *D* coincide.

EUCLID. I noticed that you did not explicitly mention the common notions that you used in your proof. I assume that this is in order, but in my opinion you leaned rather heavily on your figure. Are you sure that your reasoning would have been the same if you had drawn another figure? (*Euclid draws a third figure.*)

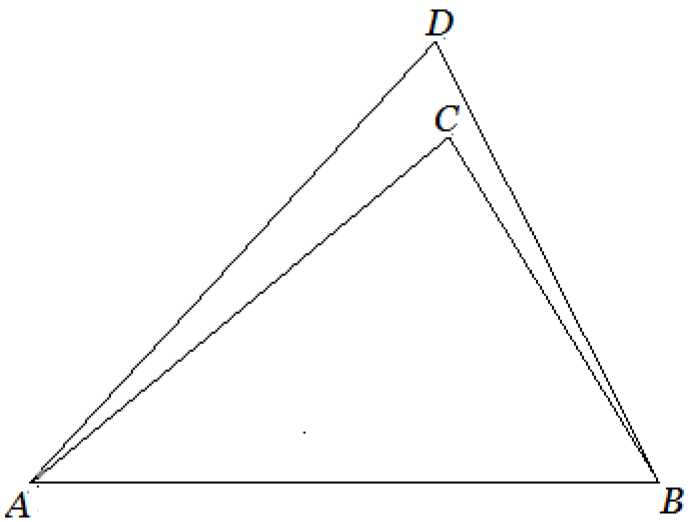

Figure 19

ARISTARCHOS. Good heavens! My reasoning does not apply to this figure. (*He pauses a moment.*) Oh, but this is nothing else than the second case, for now the vertex of one of the triangles lies inside the other triangle!

EUCLID. (*laughing*) I am glad that you saw it. Now I am not talking anymore as an Athenian philosopher, but I think that it is good that I made my last remark. Otherwise you could fall into a trap, as soon as you had to admit that your proof depended in some way or another on your figure.

ARISTARCHOS. I will now try to reproduce your proof of the second case. But tell me first, how you found is, because it requires more ingenuity than the proof of the first case.

EUCLID. I must admit that I found my proof more or less by luck. I remember the moment that I connected not only the points *C* and *D*, but also produced *BC* and *BD*, and the joy it gave, for it completed my task.

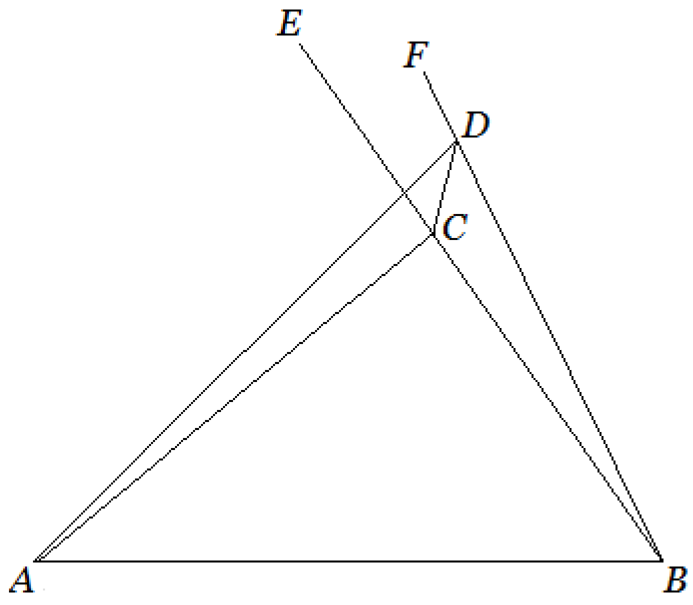

Figure 20

ARISTARCHOS. I missed this part of your proof when I read book one, but I see how it goes, thanks to your auxiliary lines, of course. The angle *ECD* is smaller than the angle *ADC*, because it is a part of the angle *ACD*, and it is much smaller than the angle *CDF*, because this angle contains the angle *ACD* as a part. But the angles *ECD* and *CDF* are equal according to the second part of proposition 5. Very nice!

EUCLID (*erases the lines CD, CE and DF in his last figure, so that the preceding one reappears*). Don't you notice anything?

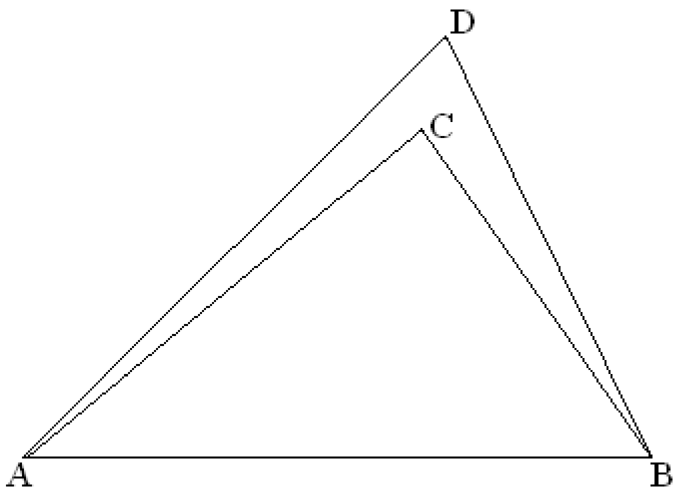

Figure 21

ARISTARCHOS. This is the same figure as I began with! If we had had proposition 21 already at our disposal in this stage of your system, then everything would have been easy. But I learned from you that we are not philosophers who mix up everything, but proceed systematically. Thank you very much. I can now safely return to Athens.

EUCLID. Ho, ho, Aristarchos. I have been told that you also did interesting new work in mathematics yourself. I am anxious to hear more about it.

ARISTARCHOS. With pleasure, but can you first offer me a drink, for I have become pretty thirsty after your lecture.

EUCLID. I am glad that you bring this up, for I always forget to take care of my condition when I am working. (*Euclid calls his servant.*)